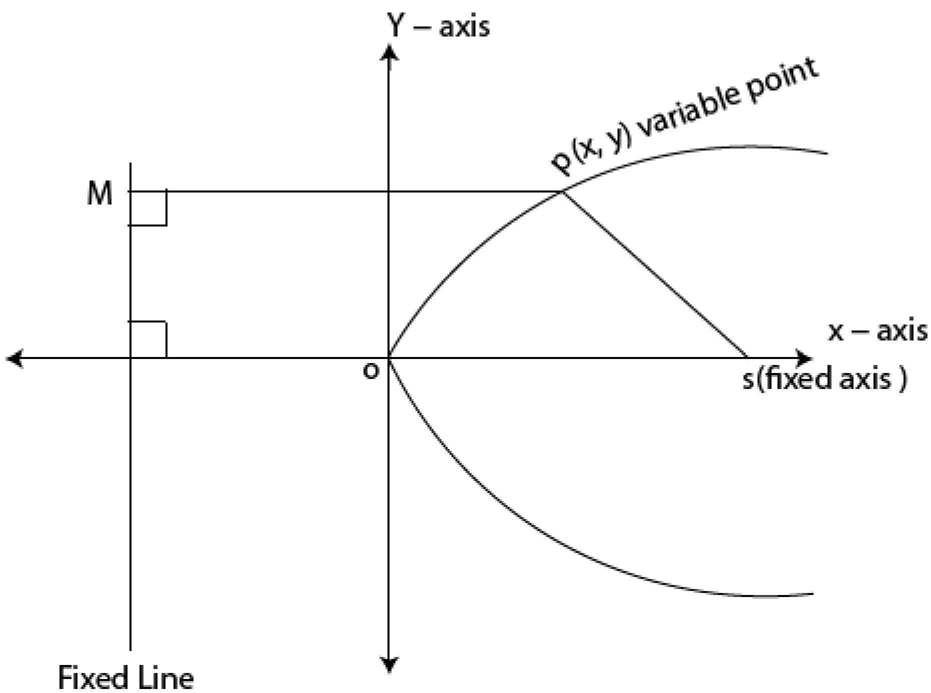
COORDINATE GEOMETRY II

CONIC SECTIONS

Definition

Conic sections or conics are the sections whose *ratios* of the distance of the variables point from the fixed point to the distance, or the variable point from the fixed line is *constant*



$$\frac{\overline{SP}}{\overline{MP}} = constant$$

<u>TYPES OF CONIC SECTION</u>

There are i) Parabola

ii) Ellipse

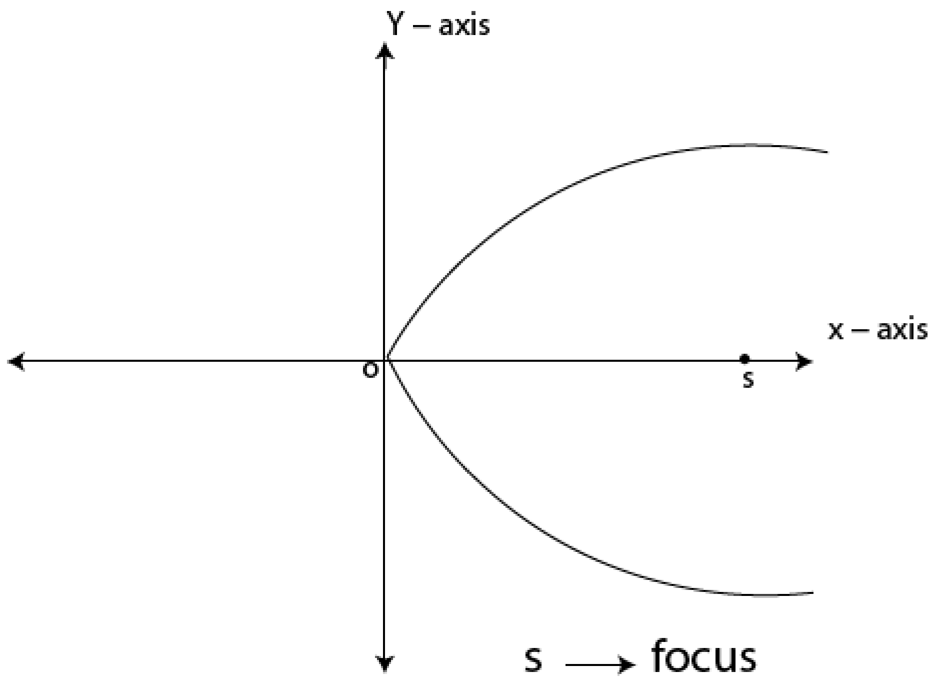
iii) Hyperbola

IMPORTANT TERMS USED IN CONIC SECTION

I. FOCUS

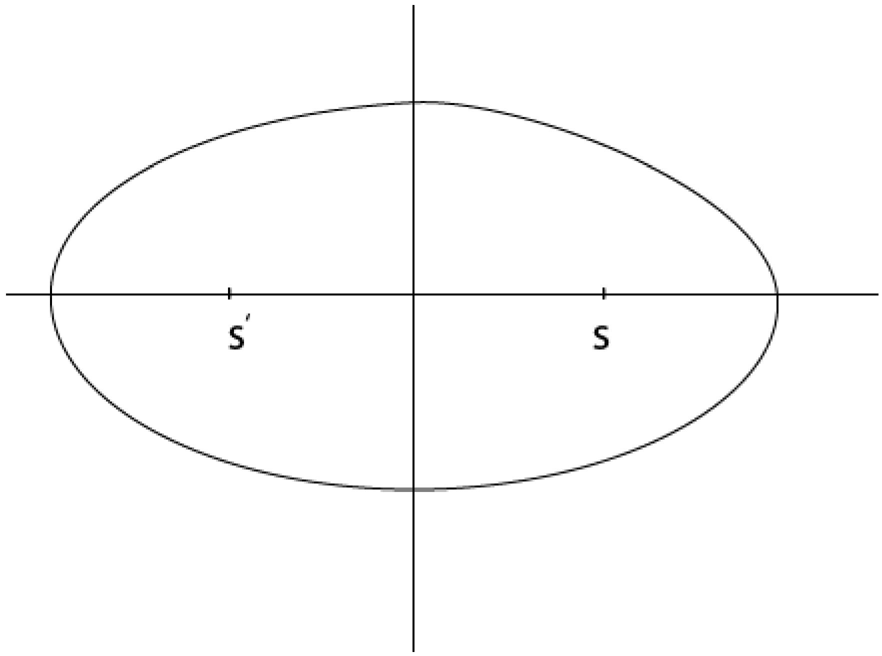
This is the *fixed point* of the conic section.

For parabola



S → focus

For ellipse

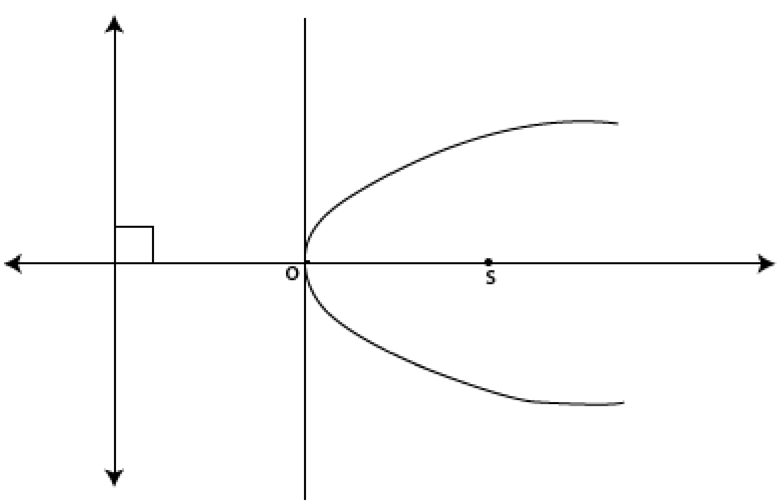


S and S’ are the foci of an ellipse

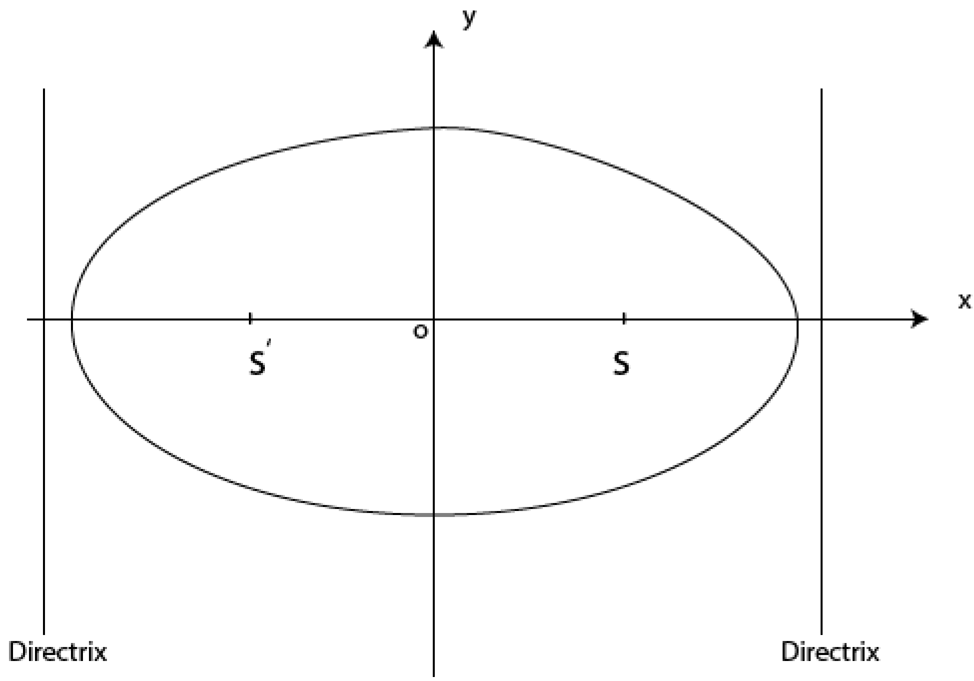
II. <u>DIRECTRIX</u>

This is the *straight line* whose *distance* from the focus is *fixed.*

<u>For parabola</u>



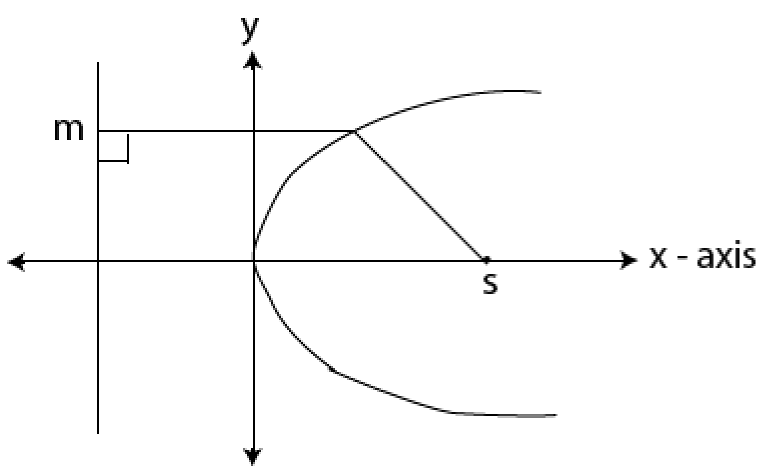
For ellipse



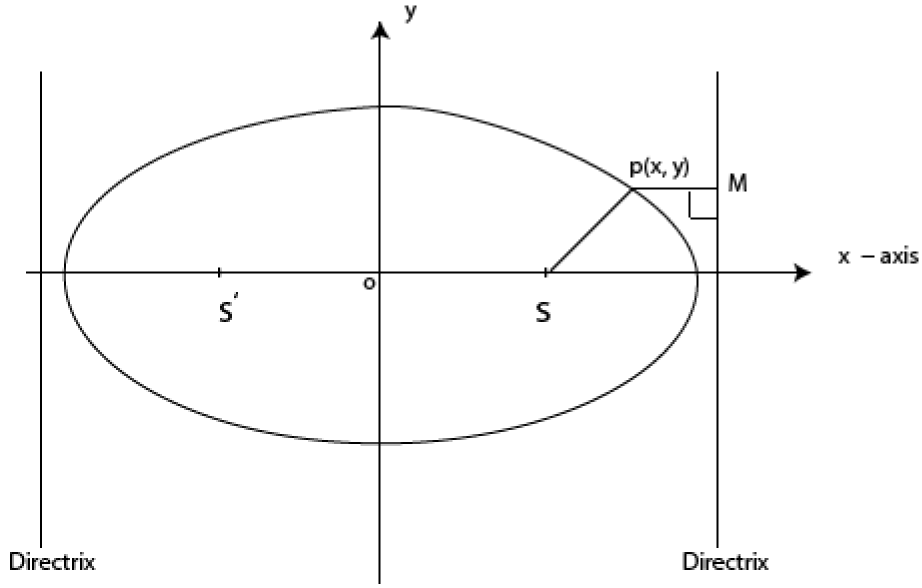
III. ECCENTRICITY (e)

This is the *amount ratio* of the *distance of the variable point from the focus* to the *distance of the variables point from the directrix.*

For Parabola



For ellipse



$$\frac{SP}{MP} = constant = e$$

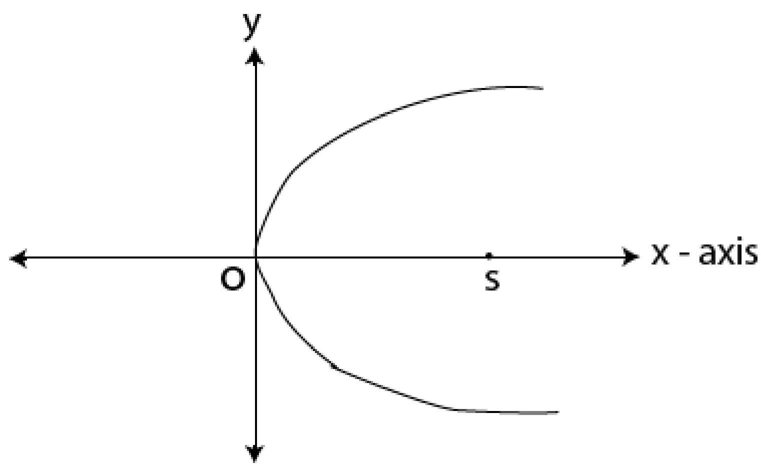
$$\frac{\overline{SP}}{\overline{MP}} = e$$

$$\frac{SP}{MP} = Constant = e$$

IV. AXIS OF THE CONIC

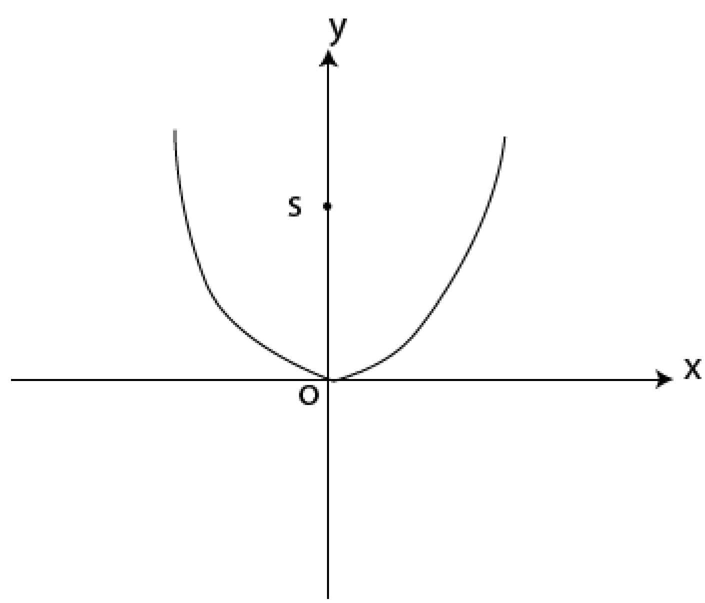
This is *the straight line* which *cuts the conic* or conic section *symmetrically* into two equal parts.

For parabola

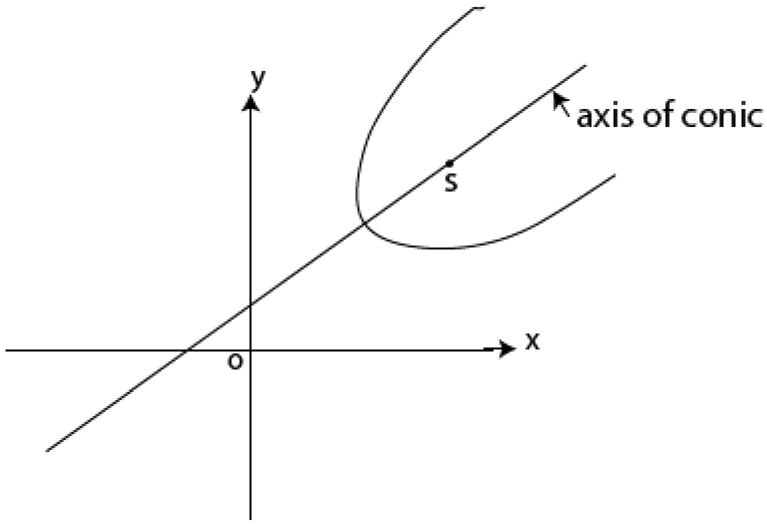


X-axis is the point of the conic i.e. parabola

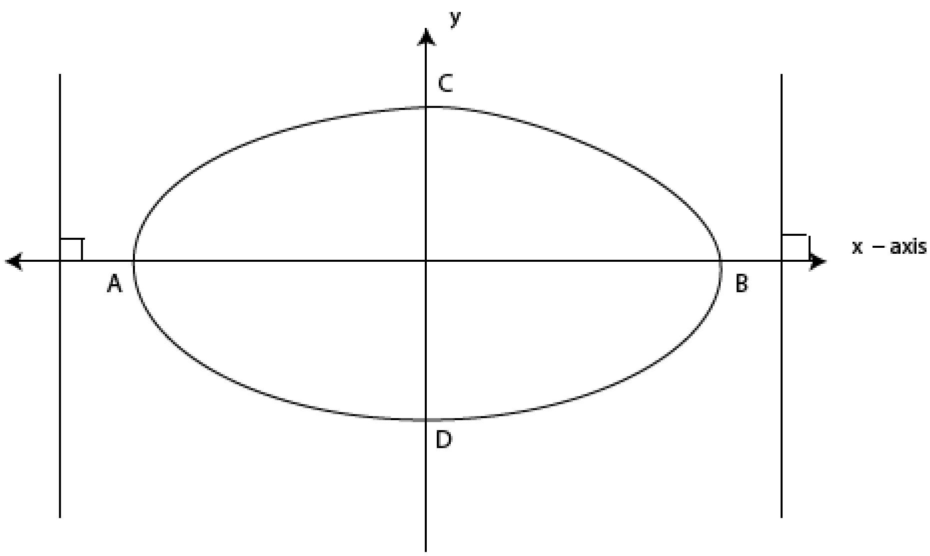
Also



Y-axis is the axis of the conic



FOR ELLIPSE



AB – is the axis (major axis) of the Conic i.e. (ellipse)

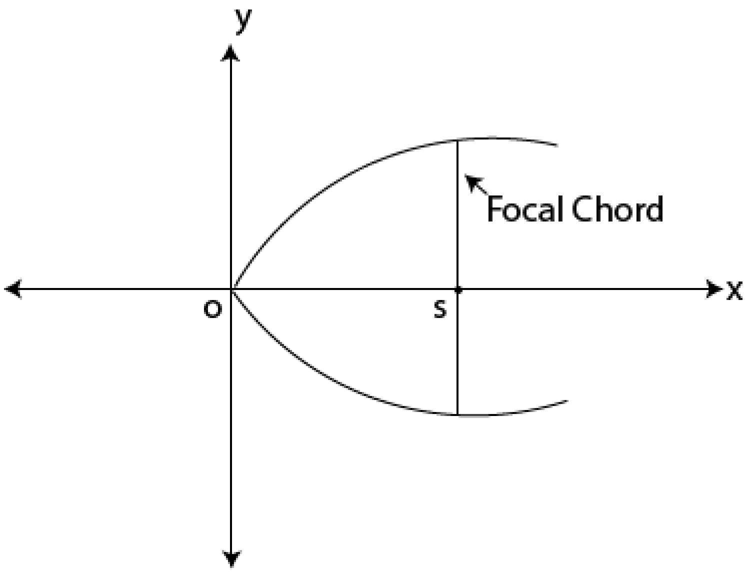
CD – is the axis (minor axis) the Conic i.e. (ellipse)

→An ellipse has Two axes is major and minor axes

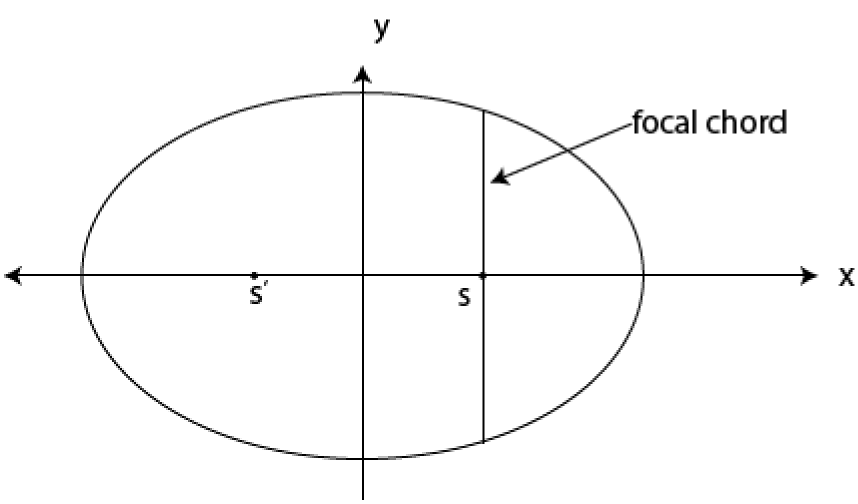
V FOCAL CHORD

This is the chord *passing through the focus* of the conic section.

For parabola



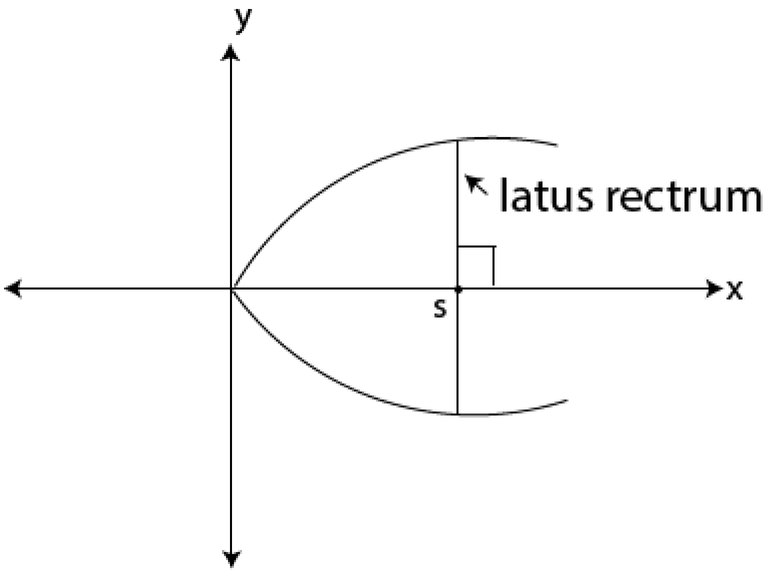
For ellipse



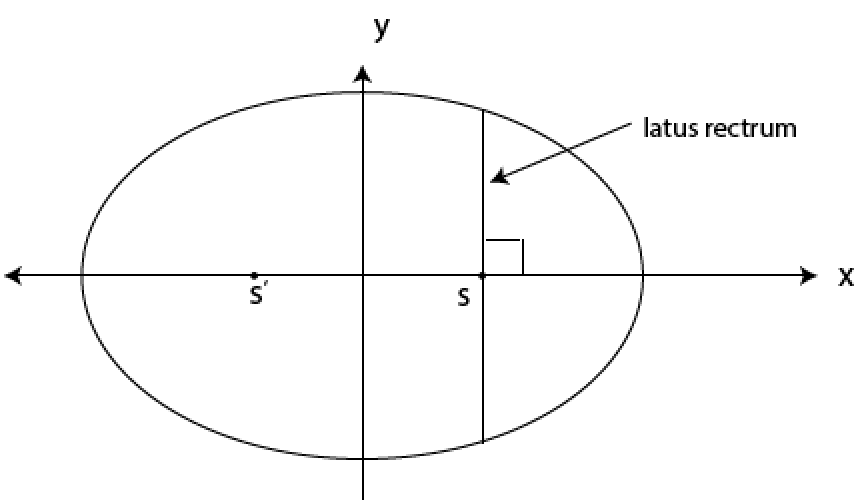
VI LATUS RECTRUM

This is the focal *cord* which is *perpendicular* to the axis of the conic section.

For parabola



For Ellipse



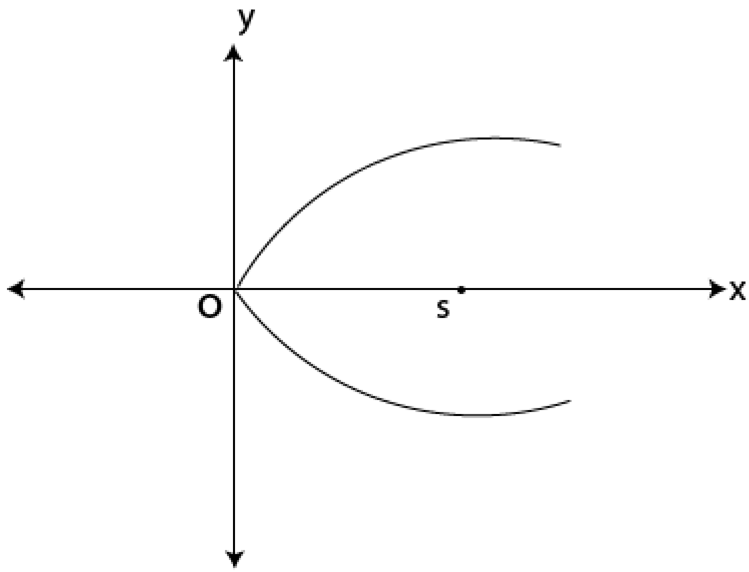
Note:

Latus rectum is always parallel to the directrix

VII. VERTEX

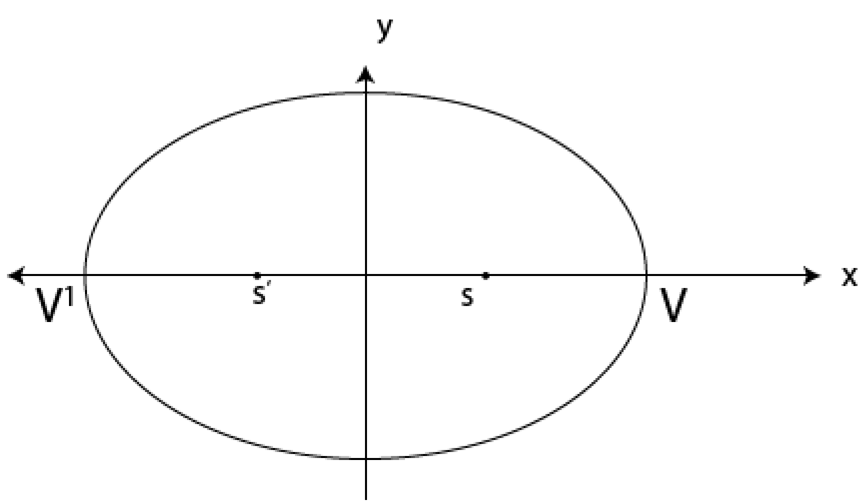
This is the *turning point* of the conic section.

For parabola



0 – is the vertex

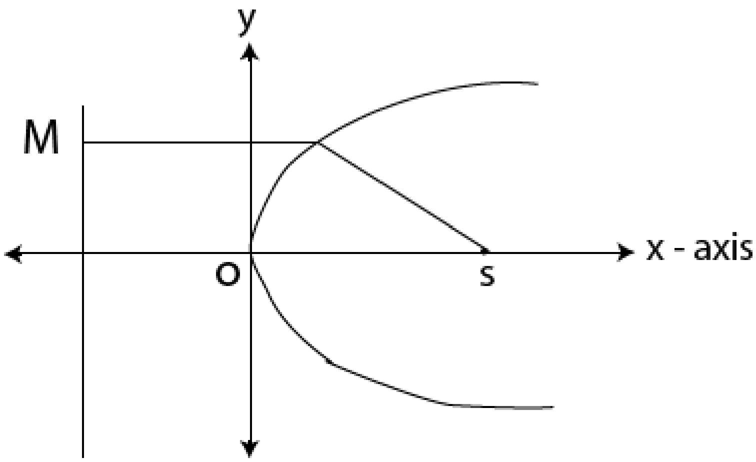
For ellipse



Where V and V^1 is the vertex of an ellipse

PARABOLA

This is the conic section whose eccentricity, e is one i.e. $e = 1$



$$\frac{SP}{MP} = e = 1$$

For parabola

$$\frac{SP}{MP} = 1$$

SP = MP

EQUATIONS OF THE PARABOLA

These are;

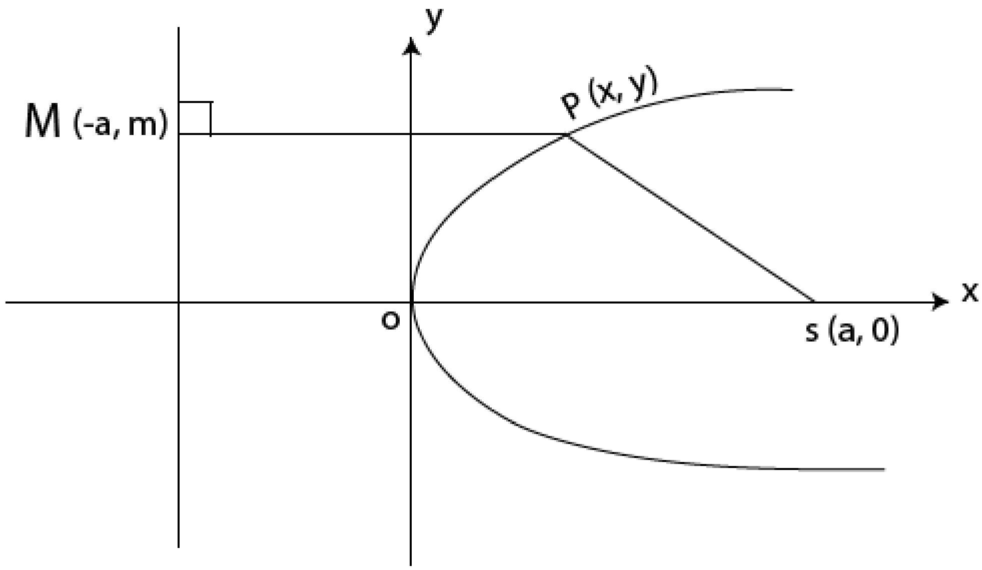
a) Standard equation

b) General equations

A. STANDARD EQUATION OF THE PARABOLA

1st case: Along the x – axis

- Consider the parabola whose focus is S(a, 0) and directrix x = -a



$\overrightarrow{SP} = \overrightarrow{MP}$

Squaring both sides

$(\overrightarrow{SP})^2 = (\overrightarrow{MP})^2$

$(x-a)^2 + (y-0)^2 = (x-a)^2 + (y-0)^2$

$(x-a)^2 + y^2 = (x+a)^2$

$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

$-2ax + y^2 = 2ax$

$y^2 = 4ax$

$\therefore y^2 = 4ax$

Is the standard equation of the parabola

PROPERTIES

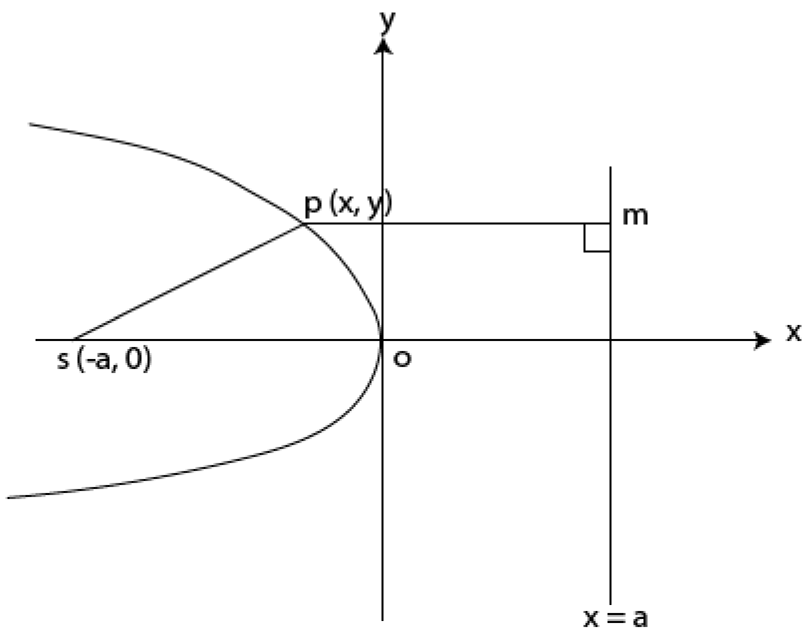
i) The parabola lies along x – axis

ii) Focus, s (a, o)

iii) Directrix x = -a

iv) Vertex (0, 0) origin

Note:



PROPERTIES

1) The parabola lies along x – axis

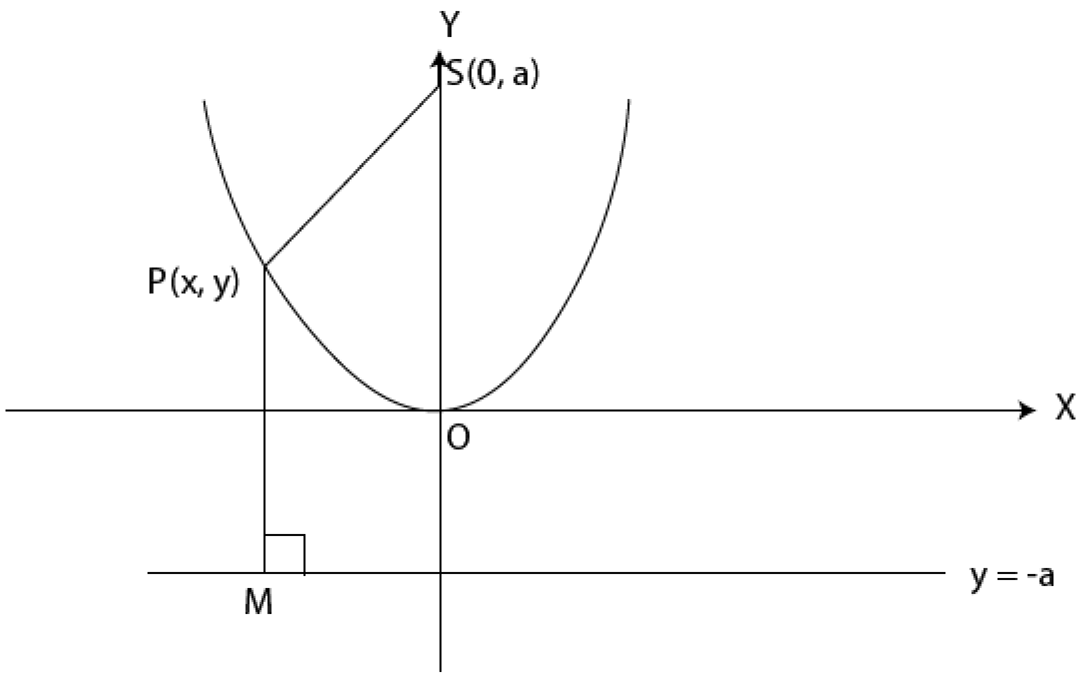
2) Focus s (-a, o)

3) Directrix x = a

4) Vertex (o, o) origin

2nd case: along y – axis

Consider the parabola when focus is s (o, a) and directrix y = -a



$\rightarrow \overline{SP} = \overline{MP}$

$(\overline{SP})^2 = (\overline{MP})^2$

$(x-0)^2 + (y-a)^2 = (x-x)^2 + (y--a)^2$

$x^2 + (y-a)^2 = 0^2 + (y+a)^2$

$x^2 + (y-a)^2 = (y+a)^2$

$x^2 - 2ay = 2ay$

$x^2 = 4ay$

$\rightarrow x^2 = 4ay$

- Is the standard equation of the parabola along y – axis

PROPERTIES

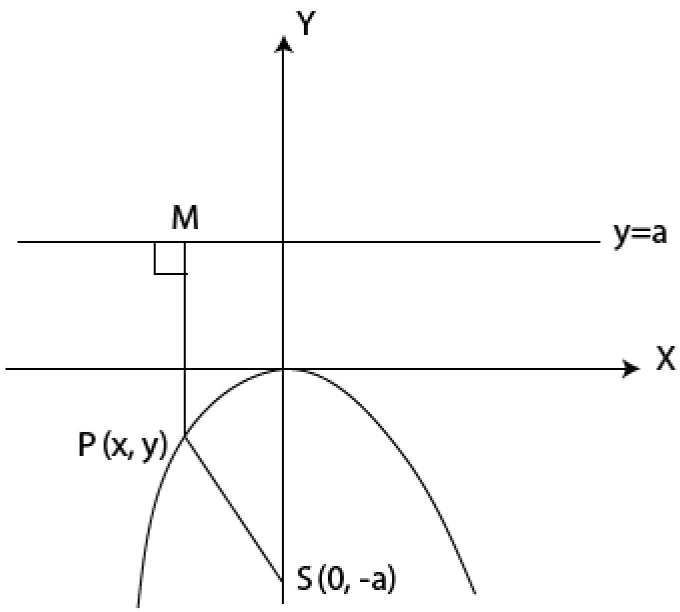
i) The parabola lies along y – axis

ii) The focus s (o, a)

iii) Directrix y = -a

iv) Vertex (o, o) origin

Note;



Hence, $x^2 = -4ay$

PROPERTIES

i) The parabola lies along y – axis

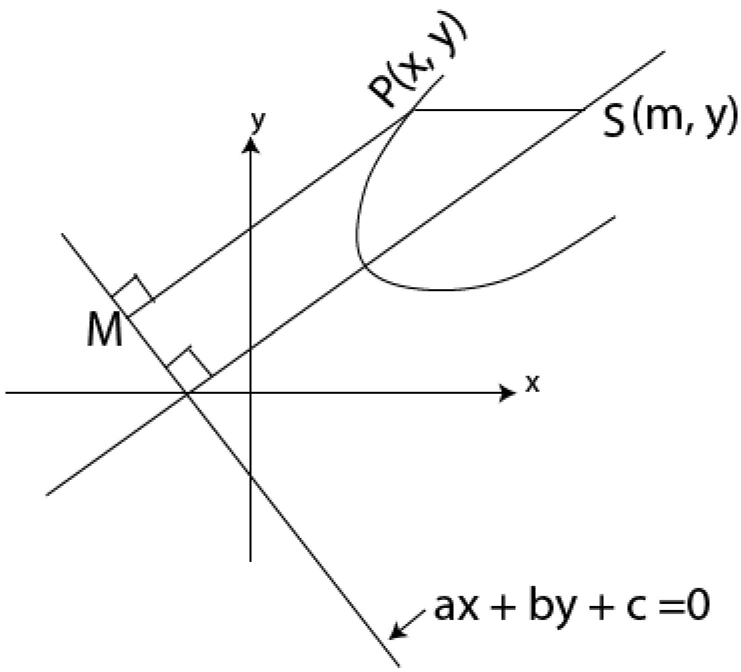
ii) Focus s (o, -a)

iii) Directrix y = a

iv) Vertex (o, o)

GENERAL EQUATION OF THE PARABOLA

- Consider the parabola whose focus is s (u, v) and directrix ax + by + c = 0



$\rightarrow \overline{SP} = \overline{MP}$

$$\sqrt{(x-u)^2 + (y-v)^2} = \left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|$$

$$(x-u)^2 + (y-v)^2 = \left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|^2$$

$$\rightarrow \boldsymbol{(x-u)^2 + (y-v)^2 = \left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|^2}$$

Is the general equation of the parabola

Where;

S (u, v) – is the focus

Examples:

1. Find the focus and directrix of the parabola $y^2 = 8x$

Solution

Given $y^2 = 8x$

Comparing from

$y^2 = 4ax$

$4ax = 8x$

$4a = 8$

$a = 2$

$focus = (a, 0)$

$= (2{,}0)$

$directrix\ x = -a$

$x = -2$

2. Find the focus and the directrix of the parabola

$y^2 = -2x$

Solution

$y^2 = -2x$

Compare with

$y^2 = -4ax$

$-4ax = -2x$

$-4a = -2$

$a = \frac{1}{2}$

$focus = (-a, 0)$

$= \left(-\frac{1}{2}, 0\right)$

$directrix\ x = a$

$x = \pm\frac{1}{2}$

3. Find the focus and directrix of $x^2 = 4y$

Solution

$x^2 = 4y$

$compare\ with$

$x^2 = 4ay$

$4a = 4$

$a = 1$

$focus = (0, a)$

$= (0,1)$

$directrix\ y = -a$

$y = -1$

4. Given the parabola $x^2 = -\frac{1}{2}y$

a) Find i) focus

ii) Directrix

iii) Vertex

b) Sketch the curve

Solution

$$x^2 = -\frac{1}{2}y$$

$$compare\ with$$

$$x^2 = -4ay$$

$$-4ay = -\frac{1}{2}y$$

$$a = \pm\frac{1}{8}$$

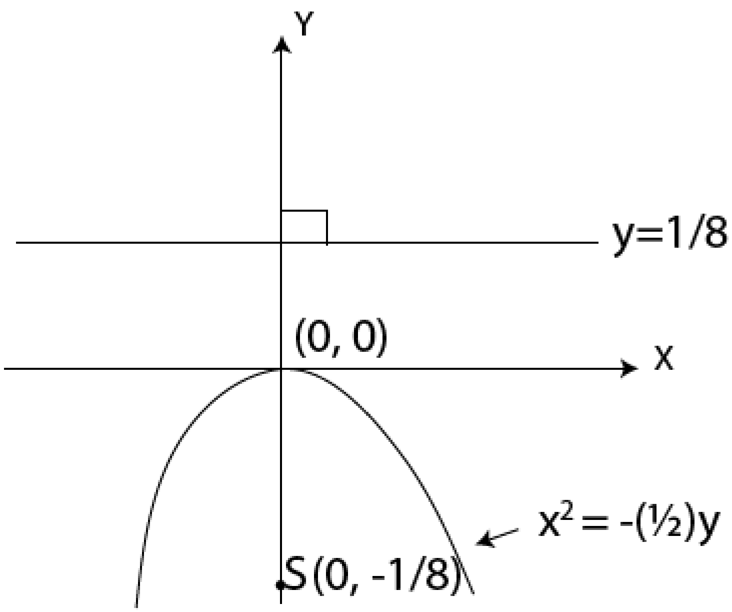
i) Focus = $(0, -a) = \left(0, -\frac{1}{8}\right)$

ii) Directrix, y = a

$$y = \frac{1}{8}$$

iii) Vertex $(0,0)$

b) Curve sketching



5. Find the equation of the parabola whose focus is (3, 0) and directrix

X = -3

Solution

Given focus (3, 0)

Directrix, x = -3

(3, 0) = (a, 0)

From

$$y^2 = 4ax$$

$y^2 = 4(3)x$

$$y^2 = 12x$$

6. Find the equation of the parabola whose directrix, y = $\frac{1}{2}$

<u>Solution</u>

Given directrix, y = $\frac{1}{2}$

Comparing with

$y = a$

$a = \frac{1}{2}$

$from\ x^2 = -4ay$

$since\ focus = (0, -a)$

$\rightarrow x^2 = -4\left(\frac{1}{2}\right)y$

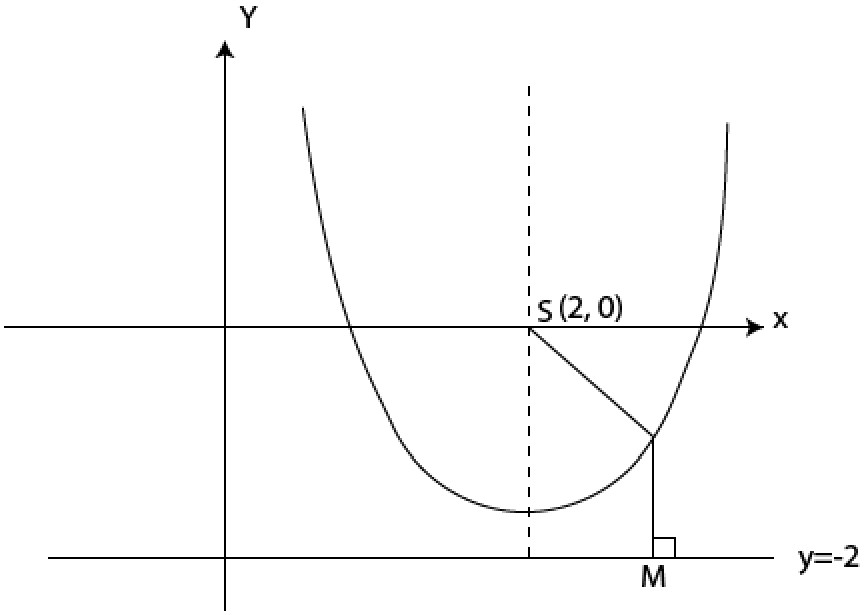
$x^2 = -2y$

7. Find the equation of the parabola whose focus is (2, 0) and directrix, y = - 2.

<u>Solution</u>

Focus = (2, 0)

Directrix y = -2



$\overline{SP} = \overline{MP}$

$(\overline{SP})^2 = (\overline{MP})^2$

$(x-2)^2 + (y-0)^2 = (x-x)^2 + (y—2)^2$

$(x-2)^2 + y^2 = y^2 + 4y + 4$

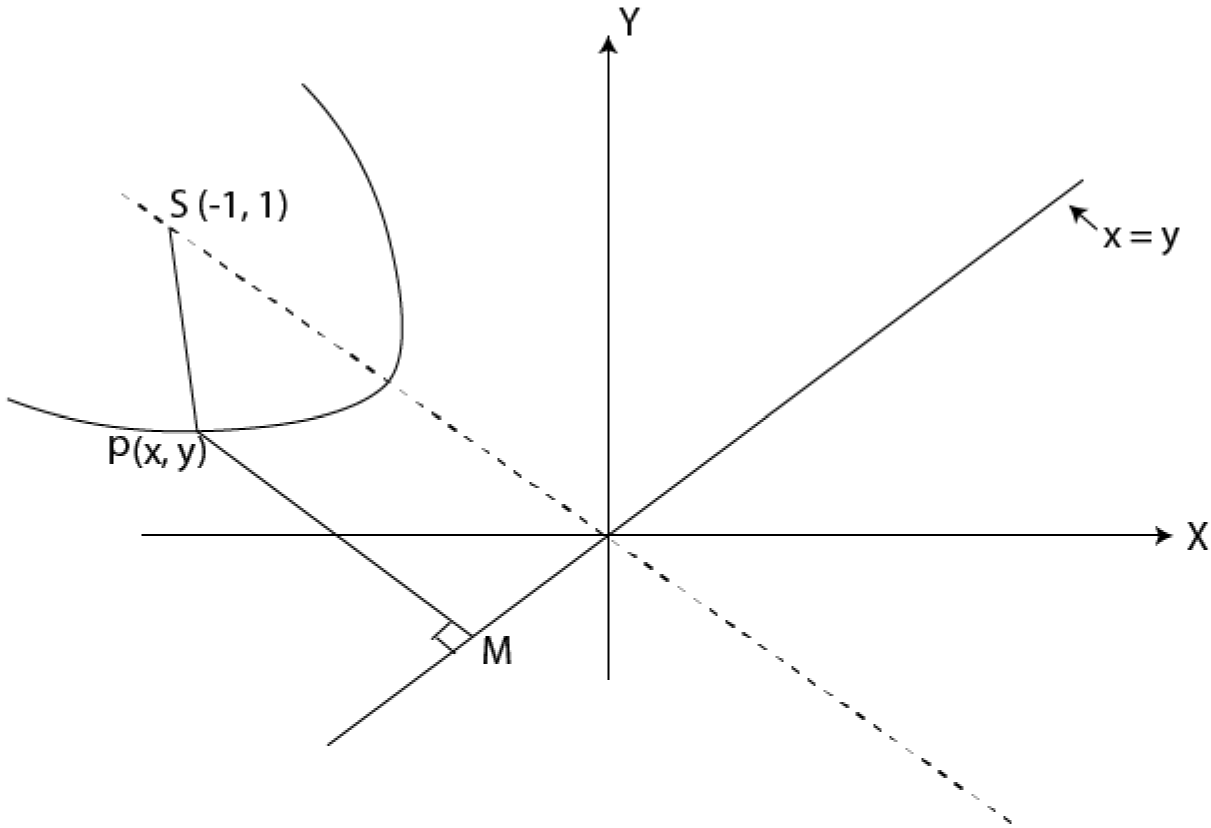
$(x-2)^2 = 4(y+1)$

8. Find the equation of the parabola whose focus is (-1, 1) and directrix x = y

<u>Solution</u>

Given: focus = (-1, 1)

Directrix x = y



$\overline{SP} = \overline{MP}$

$$\sqrt{(x - -1)^2 + (y - 1)^2} = \left|\frac{x - y}{\sqrt{(x+1)^2 + (y-1)^2}}\right|$$

$$(x + 1)^2 + (y - 1)^2 = \left|\frac{x - y}{\sqrt{2}}\right|^2$$

$$(x + 1)^2 + (y - 1)^2 = \frac{(x - y)^2}{2}$$

$$2((x + 1)^2 + (y - 1)^2) = (x - y)^2$$

$$2(x^2 + 2x + 1 + y^2 - 2y + 1) = x^2 + y^2 - 2xy$$

$$2x^2 + 2y^2 + 4x - 4y + 4 = x^2 + y^2 - 2xy$$

$$x^2 + y^2 + 4x - 4y + 2xy + 4 = 0$$

PARAMETRIC EQUATION OF THE PARABOLA

The parametric equation of the parabola are given

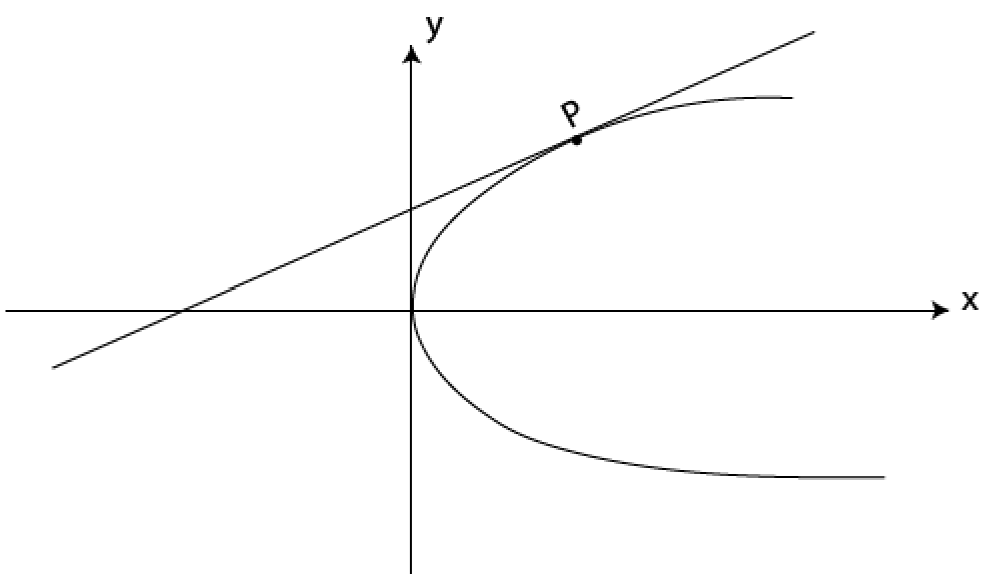
$X = at^2$ and $y = 2at$

Where;

t – is a parameter

TANGENT TO THE PARABOLA

Tangent to the *parabola*, is the *straight line* which touches it at only *one point.*



Where, p – is the point of tangent or contact

CONDITIONS FOR TANGENT TO THE PARABOLA

a) Consider a line $y = mx + c$ is the tangent to the parabola $y^2 = 4ax^2$. Hence the condition for tangency is obtained is as follows;

i.e.

$y^2 = 4ax$

$(mx + c)^2 = 4ax$

$m^2x^2 + 2mcx + c^2 = 4ax$

$(m^2)x^2 + (2mc - 4a)x + c^2 = 0$

$generaly$

$$x = \frac{-(2mc - 4a) \pm \sqrt{2m(-4a)^2 - 4(m)^2(c^2)}}{2m^2}$$

$hence$

$discriminant = 0$

$(2mc - 4a)^2 - 4m^2c^2 = 0$

$4(2mc - 4a)^2 = 4m^2c^2$

$m^2c^2 - 2mca - 2mca + 4a^2 = m^2c^2$

$m^2c^2 - 4mca + 4a^2 = m^2c^2$

$-4mca + 4a^2 = 0$

$4a^2 = 4mca$

$a = mc$

$\therefore condition\ for\ tangency\ to\ the\ parabola\ is\ a = mc$

$also$

$a = mc$

$c = \frac{a}{m}$

$\rightarrow y = mx + c$

$y = mx + \frac{a}{m}$

b) Consider the line ax + by + c = is a tangent to the parabola $y^2 = 4ax$ Hence, the condition for tangency is obtained as follows;

i.e.

$y^2 = 4ax$

$ax + by + c = 0$

$x = \frac{-by}{a} - \frac{c}{a}$

$\rightarrow y^2 = 4ax$

$y^2 = 4a\left(\frac{-by}{a} - \frac{c}{a}\right)$

$y^2 = -4by - 4c$

$y^2 + 4by + 4c = 0$

$\rightarrow this\ obeys\ the\ conditionb\ that$

$B^2 = 4AC$

$(4b)^2 = 4(1)(4c)$

$16b^2 = 16c$

$b^2 = c \rightarrow this\ is\ the\ condition\ for\ tangency\ to\ the\ parabola$

Examples

1. Prove that the parametric equation of the parabola are given by

    X = at^2, and y = 2at

    Solution

    Consider the line

    Y = mx + c is a tangent to the parabola y^2 = 4ax. Hence the condition for tangency is given by y^2 = 4ax

$(mx + c)^2 = 4ax$

$m^2x^2 + 2mcx + c^2 = 4ax$

$m^2x^2 + (2mc - 4a)x + c^2 = 0$

$generally$

$$x = \frac{-(2mc - 4a) \pm \sqrt{(2mc - 4a)^2 - 4m^2c^2}}{2m^2}$$

$$x = \frac{-(2mc - 4a) \pm \sqrt{0}}{2m^2}$$

$since\ a = mc$

$$x = \frac{-(2mc - 4a)}{2m^2}$$

$$x = \frac{-2mc + 4a}{2m^2}$$

$but\ a = mc$

$$c = \frac{a}{m}$$

$$x = \frac{-2m\left(\frac{a}{m}\right) + 4a}{2m^2}$$

$$x = \frac{-2a + 4a}{2m^2}$$

$$x = \frac{2a}{2m^2} \rightarrow \frac{a}{m^2}$$

$$x = a\left(\frac{1}{m}\right)^2$$

$let\ \frac{1}{m} = t = parameter$

$$x = at^2$$

$from$

$$y = mx + c$$

$$y = mx + \frac{a}{m}$$

$$y = m\left(\frac{a}{m^2}\right) + \frac{a}{m}$$

$$y = \frac{a}{m} + \frac{a}{m}$$

$$y = \frac{2a}{m}$$

$$y = 2a\left(\frac{1}{m}\right)$$

$$y = 2a$$

The *parametric equation* of the parabola of m is given as x = at^2 and y = 2at

Where;

t – is a parameter

GRADIENT OF TANGENT OF THE PARABOLA

The gradient of tangent to the parabola can be expressed into;

i) Cartesian form

ii) Parametric form

i) IN CARTESIAN FORM

- Consider the tangent to the parabola y^2 = 4ax Hence, from the theory.

Gradient of the curve at any = gradient of tangent to the curve at the point

$$y^2 = 4ax$$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$m = \frac{dy}{dx} = gradient\ of\ tangent\ in\ cartesian\ form$$

$$m = \frac{2a}{y}$$

ii) <u>IN PARAMETRIC FORM</u>

Consider the parametric equations of the parabola

i.e.

$$x = at^2, y = 2at$$

$$\rightarrow x = at^2$$

$$\frac{dx}{dt} = 2at$$

$$y = 2at$$

$$\frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{2a}{2at}$$

$$\frac{dy}{dx} = \frac{1}{t}$$

$$m = \frac{dy}{dx} = gradient\ of\ tangent\ in\ parametric\ form$$

$$m = \frac{1}{t}$$

EQUATION OF TANGENTS TO THE PARABOLA

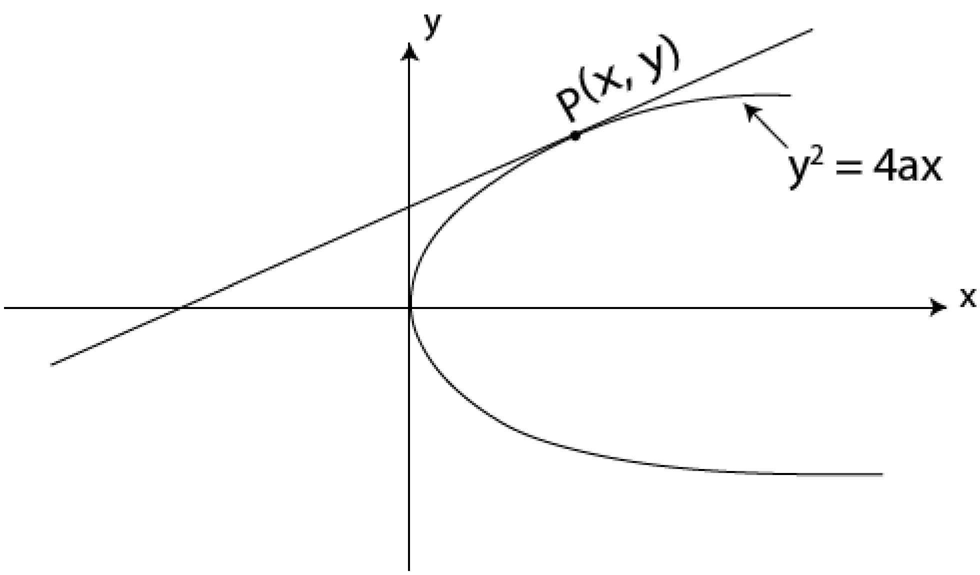
These can be expressed into;

i) Cartesian form

ii) Parametric form

i) In Cartesian form

- Consider the tangent to the parabola $y^2 = 4ax$ at the point p (x, y)



Hence the equation of tangent is given by

$$m = \frac{2a}{y_1} = \frac{y - y_1}{x - x_1}$$

$$\frac{2a}{y_1} = \frac{y - y_1}{x - x_1}$$

$$y_1(y - y_1 = 2a(x - x_1)$$

$$y_1 y - y^2{}_1 = 2ax - 2ax_1$$

$$but\ y^2 = 4ax$$

$$y^2{}_1 = 4ax_1$$

$$y_1 y - 4ax_1 = 2ax - 2ax_1$$

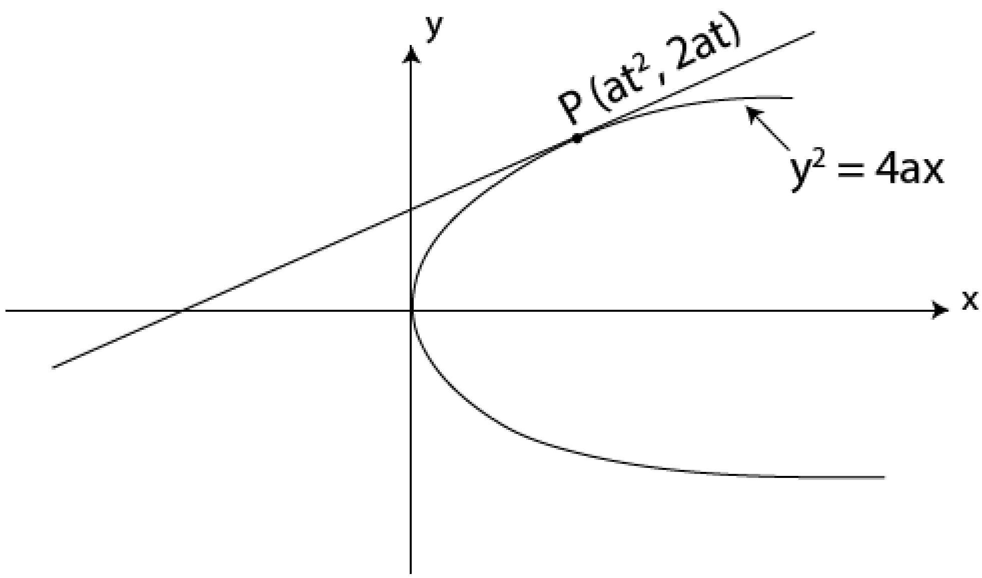
$$0 = 2ax - y_1 y - 2ax_1 + 4ax_1$$

$\therefore 2ax - y_1 y + 2ax_1 = 0$

is the equation of tangent in cartesian form

ii) In parametric form

- Consider the tangent to the parabola $y^2 = 4ax$ at the point p $(at^2, 2at)$



Hence the equation of tangent is given by;

$$m = \frac{1}{t} = \frac{y - 2at}{x - at^2}$$

$(x - at^2) = t(y - 2at)$

$x - at^2 = ty - 2at^2$

$x - ty - at^2 + 2at^2 = 0$

$x - ty + at^2 = 0$

is the equation of tangent in parametric form

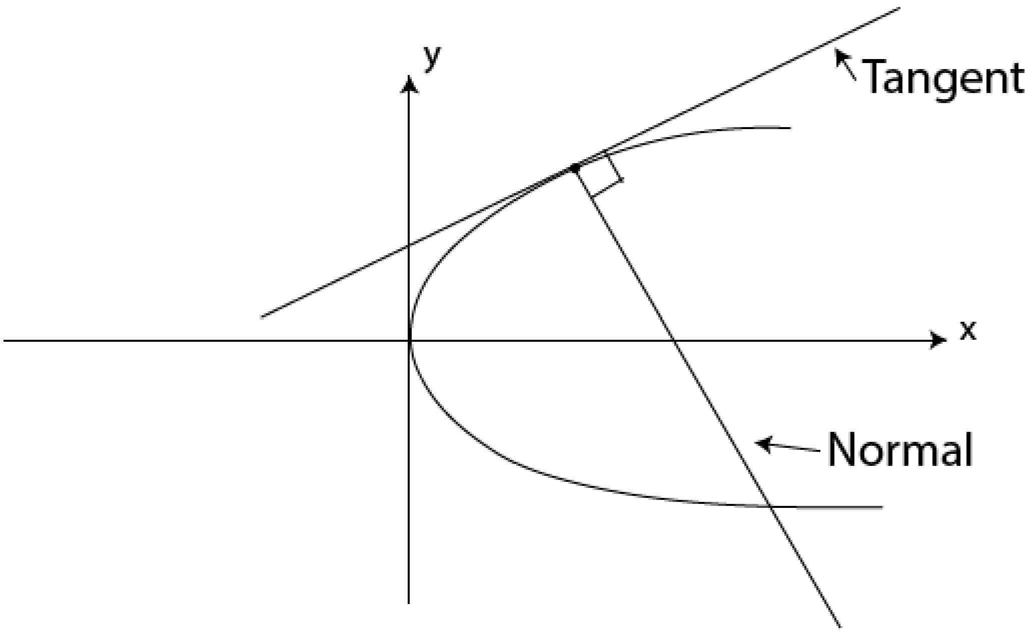
Examples

1. Show that the equation of tangent to the parabola $y^2 = 4ax$ at the point xx_1 is $yy_1 = 2a(x + x_1)$

2. Find the equation of tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$

NORMAL TO THE PARABOLA

Normal to the parabola is the line which is perpendicular at the point of tangency.



Where;

P is the point of tangency

GRADIENT OF THE NORMAL TO THE PARABOLA

This can be expressed into;·

i) Cartesian form

ii) Parametric form

i) <u>In Cartesian form</u>

- Consider the gradient of tangency in Cartesian form

i.e. $m_1 = \frac{2a}{y}$

Let M = be gradient of the normal in Cartesian form but normal is perpendicular to tangent.

$\rightarrow m_1 m = -1$

$m = \frac{-1}{m_1}$

$m = \frac{-1}{\left(\frac{2a}{y}\right)}$

$\therefore m = \frac{-y}{2a}$

ii) <u>In Parametric form</u>

Consider the gradient of tangent in parametric form.

$m_1 = \frac{1}{t}$

Let *m* be gradient of the normal in parametric form.

But

Normal is perpendicular to the tangent

$m_1 m_2 = -1$

$m = \frac{-1}{m_1}$

$$m = \frac{-1}{\frac{1}{t}}$$

$$m = -t$$

EQUATION OF THE NORMAL TO THE PARABOLA

These can be expressed into;·

i) Cartesian form

ii) Parametric form

i) In Cartesian form

Consider the normal to the point $y^2 = 4ax$ at the point p (x_1, y_1) hence the equation of the normal given by;

$$m = \frac{-y_1}{2a} = \frac{y - y_1}{x - x_1}$$

$$\frac{-y_1}{2a} = \frac{y - y_1}{x - x_1}$$

$$-y(x - x_1) = 2a(y - y_1)$$

$$-y_1x + y_1x_1 = 2ay - 2ay_1$$

$$-y_1x - 2ay + y_1x_1 + 2ay_1 = 0$$

$$y_1x + 2ay - y_1(x_1 + 2a) = 0 \textit{ is cartesian normal to the parabola}$$

ii) In parametric form

Consider the normal to the parabola $y^2 = 4ax$ at the point p $(at^2, 2at)$. Hence the equation of the normal is given by;

$$m = -t = \frac{y - 2at}{x - at^2}$$

$$-t = \frac{y - 2at}{x - at^2}$$

$$-t(x - at^2) = y - 2at$$

$$-tx + at^3 = y - 2at$$

$$-tx - y + at^3 + 2at = 0$$

$$\boldsymbol{tx + y - at^3 - 2at = 0}$$

is the equation of the normal in the parametric form

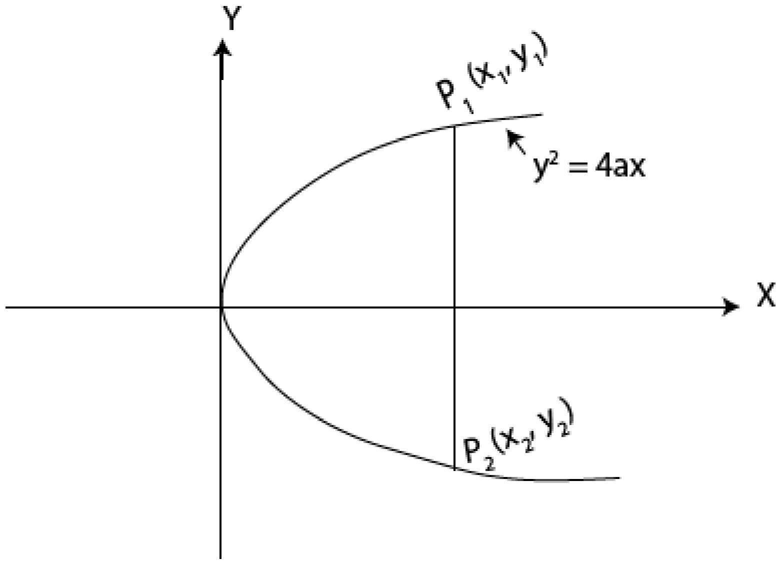
Examples:

1. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1)

2. Show that the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^3, 2at)$ is $tx + y - at^3 - 2at = 0$

CHORD TO THE PARABOLA

- This is the *line joining two points* on the parabola



Let *m* – be gradient of the chord
Hence

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

but

$$y^2 = 4ax$$

$$y^2{}_1 = 4ax_1 \ldots \ldots \ldots (i)$$

$$y_2{}^2 = 4ax_2 \ldots \ldots \ldots (ii)$$

$taking\ equation\ (ii) minus\ (i)$

$$y^2{}_2 - y^2{}_1 = 4ax_2 - 4ax_1$$

$$y^2{}_2 - y^2{}_1 = 4a(x_2 - x_1)$$

$$(y_2 - y_1)(y_2 + y_1) = 4a(x_2 - x_1)$$

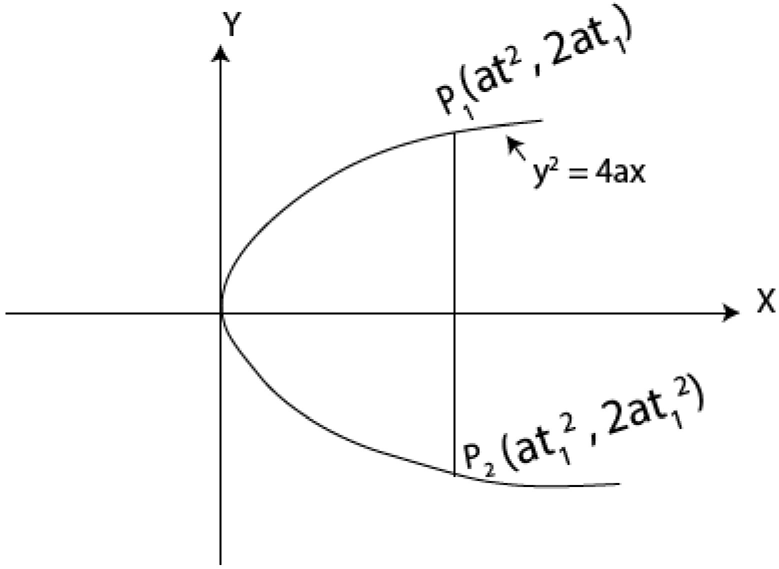
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1}$$

$$\therefore m = \frac{4a}{y_1 + y_2}$$

ii) <u>GRADIENT OF THE CHORD IN PARAMETRIC FORM</u>

Consider a chord to the parabola $y^2 = 4ax$ at the points $P_1(at^2, 2at_1)$ and $P_2(at_1{}^2, 2at_1{}^2)\ shown\ below$



$M - be\ gradient\ of\ the\ chord\ in\ parametric\ form$

$$m = \frac{2at_2 - 2at_1}{at_2{}^2 - at_1{}^2}$$

$$m = \frac{2a(t_2 - t_1)}{a(t_2 - t_1)(t_2 + t_1)}$$

$$\therefore m = \frac{2}{t_1 + t_2}$$

EQUATION OF THE CHORD TO THE PARABOLA.

These can be expressed into;·

i) Cartesian form

ii) Parametric form

i) EQUATION OF THE CHORD IN PARAMETRIC FORM

- Consider the chord to the parabola $y^2 = 4ax$ at the points $P_1(at_1{}^2, 2at_1)and\ p_2(at_2{}^2, 2at_2)$. Hence the equation of the chord is given by;

$$m = \frac{2}{t_1 + t_2} = \frac{y - 2at_1}{x - at_1{}^2}$$

$$\frac{2}{t_1 + t_2} = \frac{y - 2at_1}{x - at_1{}^2}$$

$$2(x - at_1{}^2) = (t_1 + t_2)(y - 2at_1)$$

$$2x - 2at_1{}^2 = t_1 y - 2at_1{}^2 + t_2 y - 2at_1 t_2$$

$$2x = t_1 y + t_2 y - 2at_1 t_2$$

$$2x = y(t_1 + t_2) - 2at_1 t_2$$

$$\therefore 2x - y(t_1 + t_2) + 2at_1 t_2$$

is the equation of the chord in parametric form

II. <u>EQUATION OF THE CHORD IN CARTESIAN FORM.</u>

Consider the chord to the parabola $y^2 = 4ax$ at the point $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ hence the equation of the chord is given by

$$m = \frac{4a}{y_1 + y_2} = \frac{y - y_1}{x - x_1}$$

$$\frac{4a}{y_1 + y_2} = \frac{y - y_1}{x - x_1}$$

$$4a(x - x_1) = (y_1 + y_2)(y - y_1)$$

$$4ax - 4ax_1 = y_1 y - y_1{}^2 + y_2 y - y_1 y_2$$

$$but\ y^2 = 4ax$$

$$y_1{}^2 = 4ax_1$$

$$4ax - 4ax_1 = y_1 y - 4ax_1 + y_2 y - y_1 y_2$$

$$4ax = y_1 y + y_2 y - y_1 y_2$$

$$4ax = y(y_1 + y_2) - y_1 y_2$$

$$\therefore 4ax - y(y_1 + y_2) + y_1 y_2 = 0$$

is the equation of the chord in cartesian form

<u>EXCERSICE.</u>

1. Show that equation of the chord to the parabola $y^2 = 4ax$ at (x_1, y_1) and (x_2, y_2) is

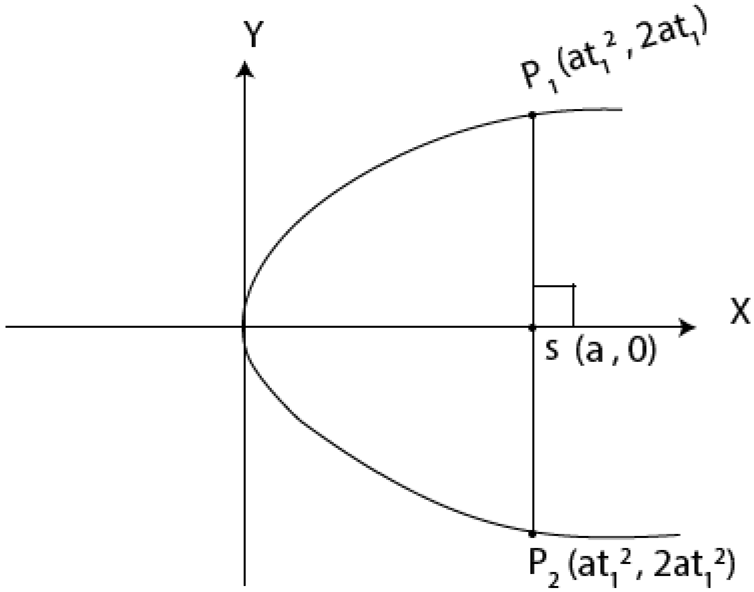
   $$4ax - y(y_1 + y_2) + y_1 y_2 = 0$$

2. Find the equation of the chord joining the points $(at_1{}^2, 2at_1)$ and $(at_1{}^2, 2at_1{}^2)$

3. As $t_2 \to t_1$, the chord approaches the tangent at t_1. deduce the equation of the tangent from the equation of the chord to the parabola $y^2 = 4ax$.

THE LENGTH OF LATUS RECTUM

Consider the parabola $y^2 = 4ax$



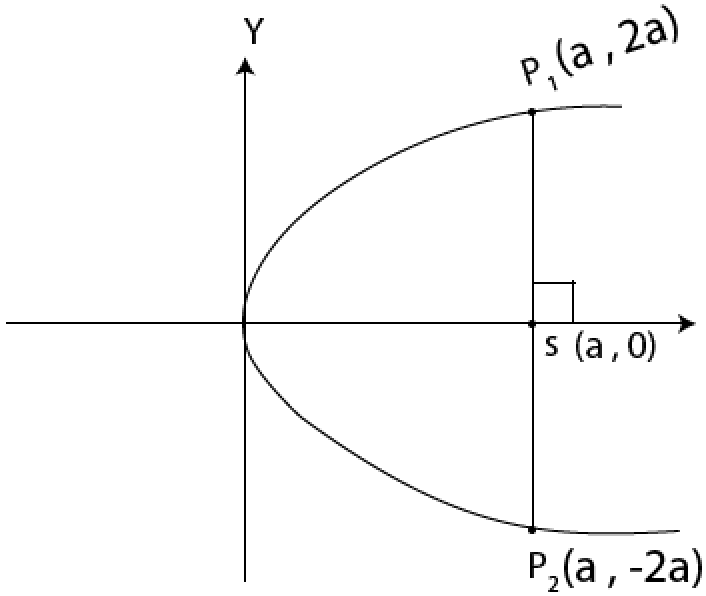
$$at_1{}^2 = a = at_2{}^2$$

$$t_1{}^2 = 1 = t_2{}^2$$

${t_1}^2 = 1, {t_2}^2 = 1$

${t_1}^2 = \pm 1, {t_2}^2 = \pm 1$

Now consider another diagram below



Therefore, the length of latus rectum is given by

$L_r = \sqrt{(a-a)^2 + (2a - -2a)^2}$

$L_r = \sqrt{0^2 + (4a)^2}$

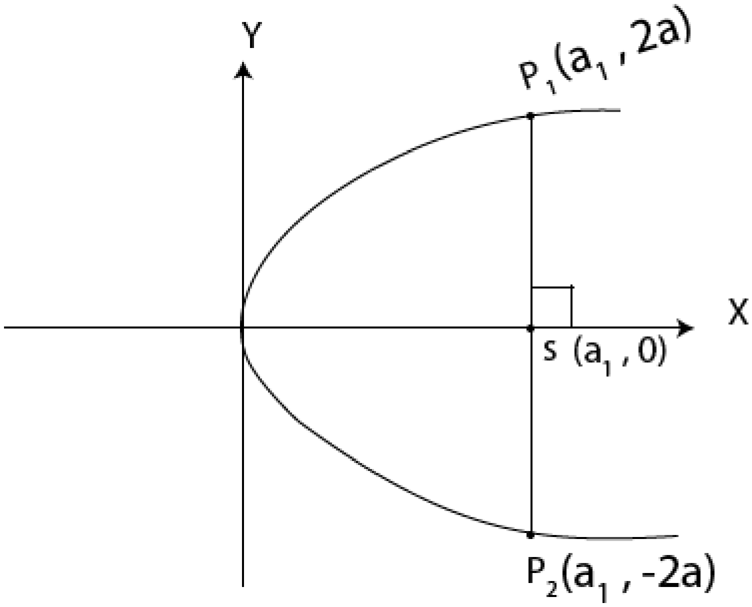
$L_r = \sqrt{(4a)^2}$

$\therefore L_r = \sqrt{(4a)^2}$

<u>EQUATION OF LATUS RECTUM</u>

- The extremities of latus rectum are the points p_1 (a, 2a) and

p_2 (a_1, -2a) as shown below



Therefore, the equation of latus rectum is given by

$$m = \frac{2a - -2a}{a - a} = \frac{y - 2a}{x - a}$$

$$\frac{2a + 2a}{0} = \frac{y - 2a}{x - a}$$

$$\frac{4a}{0} = \frac{y - 2a}{x - a}$$

$$4a(x - a) = 0(y - 2a)$$

$$x - a = 0$$

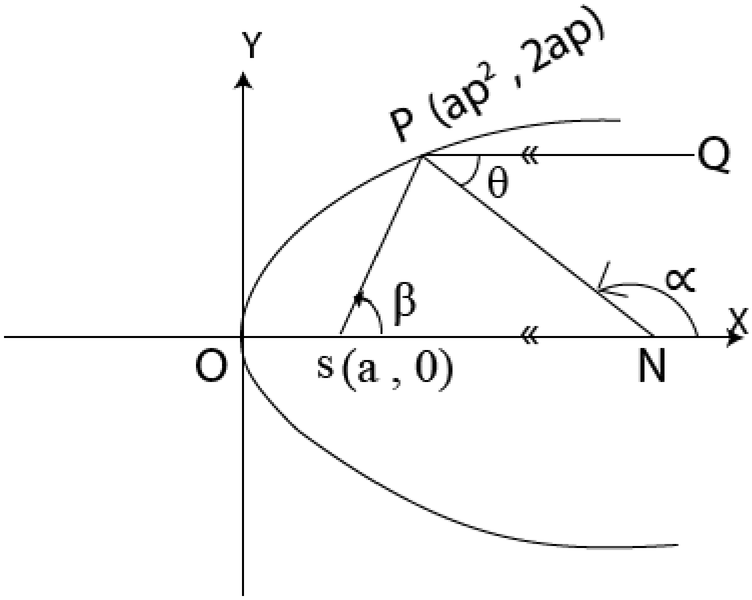
$$x = a$$

is the equation of latus rectum

OPTICAL PROPERTY OF THE PARABOLA

Any ray parallel to the axis of the parabola is reflected through the focus. This property which is of considerable practical use in optics can be proved by showing that the normal line at the point ''p'' on the parabola bisects the angle between $\overline{PS}$ and the line $\overline{PQ}$ which is parallel to the axis of the parabola.

Angle of INCIDENCE and angle of REFLECTION are equal



$\overline{PN}$ – is the normal line at the point 'p' on the parabola

i.e.

$\tan\theta = -P$

$\rightarrow \tan\theta = \tan(\pi - \alpha)$

$\tan\theta = \dfrac{\tan\pi - \tan\alpha}{1 + \tan\pi\tan\alpha}$

$\tan\theta = \dfrac{0 - \tan\alpha}{1 + 0}$

$\tan\theta = -\tan\alpha$

$\tan\theta = -(-P)$

$\tan\theta = P \ldots\ldots(i)$

$\rightarrow \tan\beta = \dfrac{2ap - 0}{ap^2 - a}$

$\rightarrow \tan\beta = \dfrac{2ap}{ap^2 - a}$

$\tan\beta = \dfrac{a(2p)}{a(p^2 - 1)}$

$\tan\beta = \dfrac{2p}{p^2 - 1}$

$\tan\beta = \dfrac{2p}{p^2 - 1}$

$but \tan\theta = P$

$\tan\beta = \dfrac{2\tan\theta}{tan^2\theta - 1}$

$\rightarrow \tan(QPS) = \tan(\pi - \beta)$

$\tan(QPS) = \dfrac{\tan\pi - \tan\beta}{1 + \tan\pi\tan\beta}$

$\tan(QPS) = \dfrac{0 - \tan\beta}{1 + 0}$

$\tan(QPS) = -\tan\beta$

$but \tan\beta = \dfrac{2\tan\theta}{tan^2\theta - 1}$

$\tan(QPS) = -\left[\dfrac{2\tan\theta}{tan^2\theta - 1}\right]$

$\tan(QPS) = \dfrac{2\tan\theta}{1 - tan^2\theta}$

$\tan(QPS) = \tan 2\theta$

$QPS = 2\theta$

$QPS = 2.OPN$ Note that; (QPS) Is an angle.

$\therefore \overline{PN} \rightarrow bisect\ the\ angle\ betwwen\ \overline{PS}and\ \overline{PQ}\ i.e\ the\ angle$

$of\ incidence\ and\ angle\ of\ reflectionare\ equal$

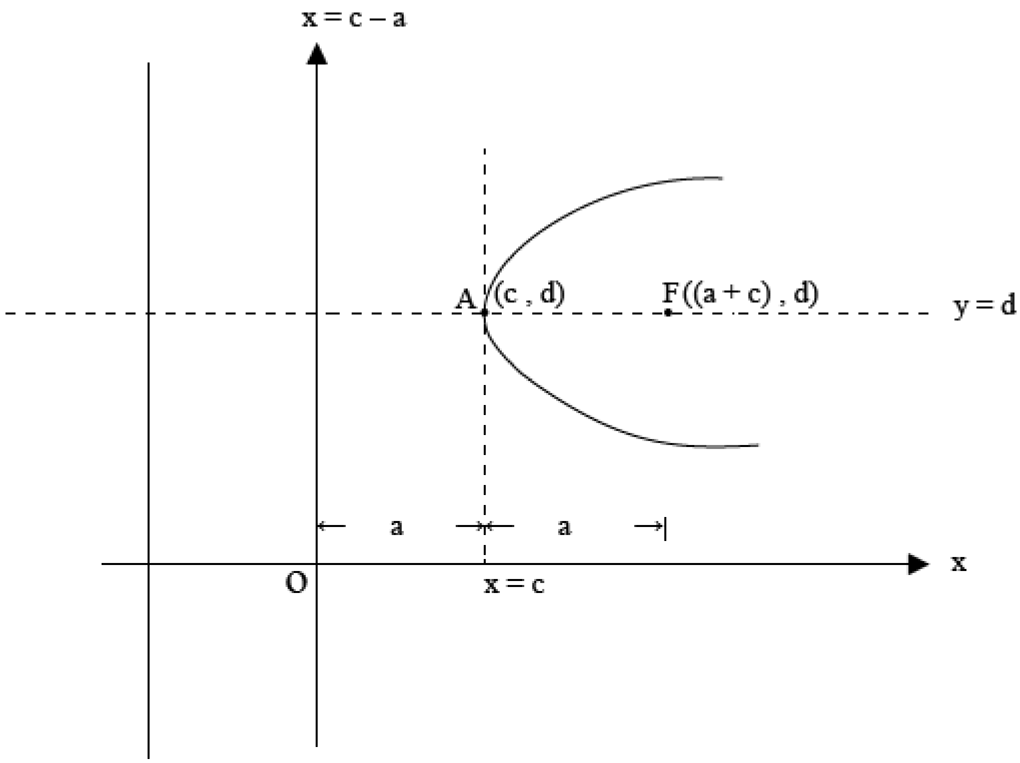
Examples

Prove that rays of height parallel to the axis of the parabolic mirror are reflected through the focus.

<u>TRANSLATED PARABOLA</u>

1. $(y-d)^2 = 4a(x-c)$

- consider the parabola below



PROPERTIES.

I) The parabola is symmetrical about the line y = d through the focus

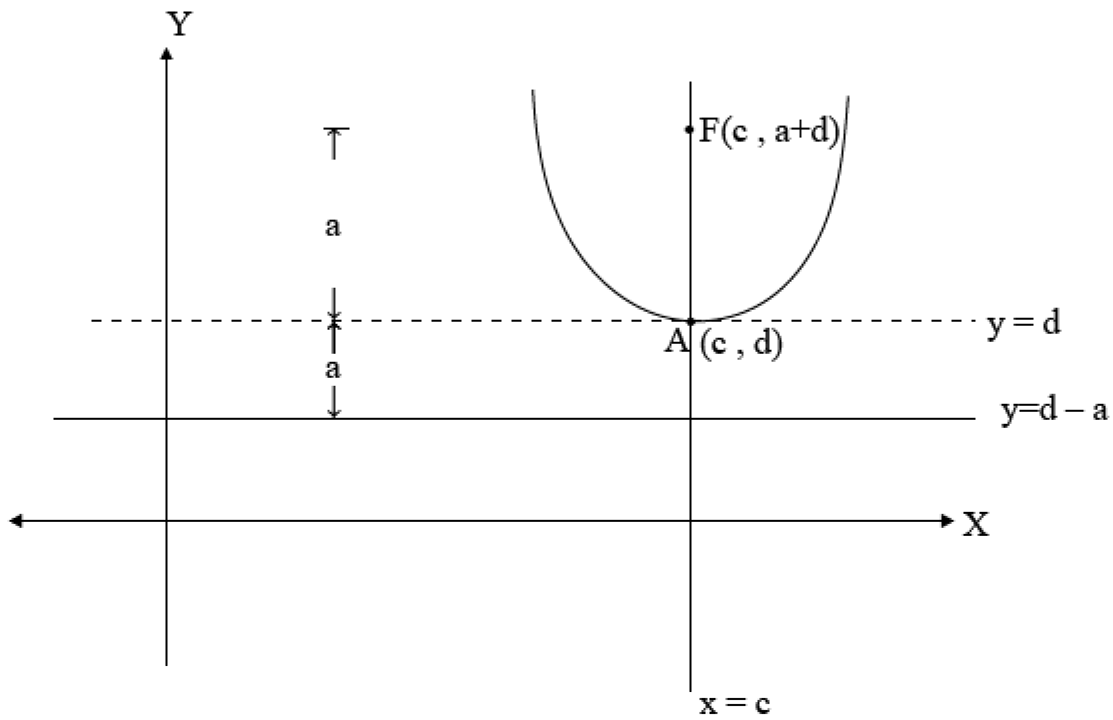
II) Focus, $F(a+c,d)$

III) Vertex, $A(c,d)$

IV) Directrix, $x=-a+c$

2. $(x-c)^2=4a(y-d)$

- Consider the parabola below



PROPERTIES

I) the parabola is symmetrical about the line x = c, through the focus

II) Focus $F(c, a+d)$

III) Vertex, $A(c, d)$

IV) Directrix, $y = -a + d$

Examples

1. Show that the equation $y = 5x - 2x^2$ represent the parabola and hence find

i) Focus

ii) Vertex

iii) Directrix

iv) Length of latus rectum

<u>Solution</u>

Given;

$$y = 5x - 2x^2$$

$$5x - 2x^2 = y$$

$$-2x^2 + 5x = y$$

$$-2\left(x^2 - \frac{5}{2}x\right) = y$$

$$-2\left(x^2 - \frac{5}{2}x + \frac{25}{16}\right) = y + \frac{25}{16}(-2)$$

$$-2\left(x - \frac{5}{4}\right)^2 = y - \frac{25}{8}$$

$$\left(x - \frac{5}{4}\right)^2 = -\frac{1}{2}\left(y - \frac{25}{8}\right)$$

$$\left(x - \frac{5}{4}\right)^2 = 4\left(-\frac{1}{8}\right)\left(y - \frac{25}{8}\right)$$

$$put\ x - \frac{5}{4} = x\ and\ y - \frac{25}{8} = y$$

$$where\ a = \frac{1}{8}.therefore, the\ equation\ represents\ a\ parabola$$

$hence\ shown$

$$\rightarrow\left(x-\frac{5}{4}\right)^2=4\left(-\frac{1}{8}\right)\left(y-\frac{25}{8}\right)$$

$comparing\ with$

$(x-c)^2=-4a(y-d)$

$c=\frac{5}{4}, d=\frac{25}{8}, a=\frac{1}{8}$

$(i) Focus$

$=(0,-a)+(c,d)$

$=(c,-a+d)$

$=\frac{5}{4}, -\frac{1}{8}+\frac{25}{8}$

$=\left(\frac{5}{4}, 3\right)$

$(ii) vertex$

$=(0,0)+(c,d)$

$=(c,d)$

$=\left(\frac{5}{4}, \frac{25}{8}\right)$

$(iii) directrix$

$y=a+d$

$y=\frac{1}{8}+\frac{25}{8}$

$y=\frac{26}{8}$

$\therefore y = \frac{13}{4}$

$(iv) length\ of\ latus\ rectum$

$from$

$L_r = |4a|$

$= \left|4\left(\frac{1}{8}\right)\right|$

$L_r = \frac{1}{2} units$

2. Shown that the equation $x^2 + 4x + 2 = y$ represents the parabola hence find its focus.

Solution

Given;

$x^2 + 4x + 2 = y$

$x^2 + 4x + 4 = y + 2$

$(x + 2)^2 = y + 2$

$(x + 2)^2 = 4\left(\frac{1}{4}\right)(y + 2)$

$put\ x + 2 = X$

$y + 2 = Y$

$X^2 = 4aY$

$where\ a = \frac{1}{4}$

$\rightarrow representys\ a\ parabola$

$hence\ shown$

$\rightarrow (x+2)^2 = 4\left(\frac{1}{4}\right)(y+2)$

$comparing\ from$

$(x-c)^2 = -4a(y-d)$

$c = -2, d = -2, a = \frac{1}{4}$

$\rightarrow Focus = (0, a) + (c, d)$

$= \left(-2, \frac{1}{4} - 2\right)$

$Focus = (-2, \frac{-7}{4})$

3. Show that the equation $x^2 + 4x - 8y - 4 = 0$ represents the parabola whose focus is at (-2, 1)

Solution

$x^2 + 4x - 8y - 4 = 0$

$x^2 + 4x = 8y + 4$

$x^2 + 4x + 4 = 8y + 8$

$(x+2)^2 = 8(y+1)$

$(x+2)^2 = 4(2)(y+1)$

$put\ x + 1 = X$

$y + 1 = Y$

$\therefore X^2 = 4ax$

$where\ a = 2$

$\rightarrow the\ equation\ represents\ a\ parabola$

$hence\ shown$

$\rightarrow (x + 2)^2 = 4(2)(y + 1)$

$compare\ from$

$(x - c)^2 = 4a(y - d)$

$c = -2, d = -1, a = 2$

$Focus = (0, a) + (c, d)$

$= (c, a + d)$

$= (-2, 2 - 1)$

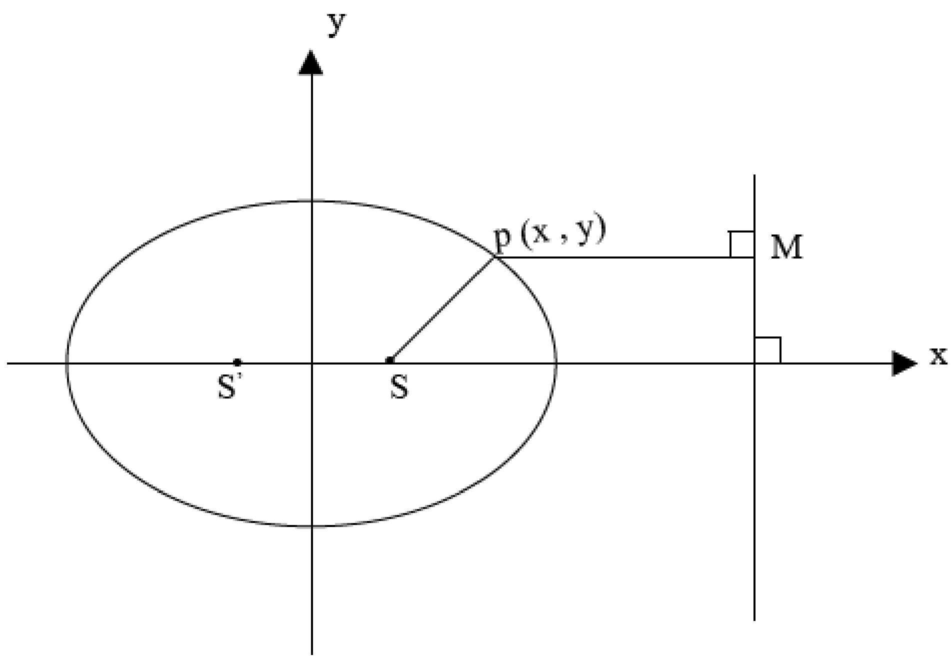
$Focus = (-2,1)$

$hence\ shown$

ELLIPSE

This is the conic section whose eccentricity e is less than one

I.e. |e| < 1



$$hence \frac{\overline{SP}}{\overline{MP}} = e \; where \; |e| < 1$$

AXES OF AN ELLIPSE

An ellipse has two axes these are
i) Major axis
ii) Minor axis

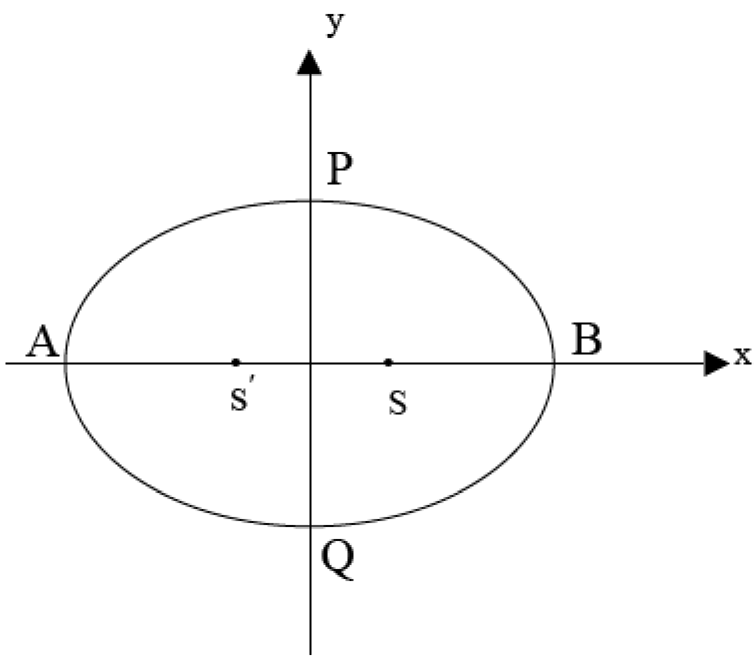
1. MAJOR AXIS

Is the one whose length is large

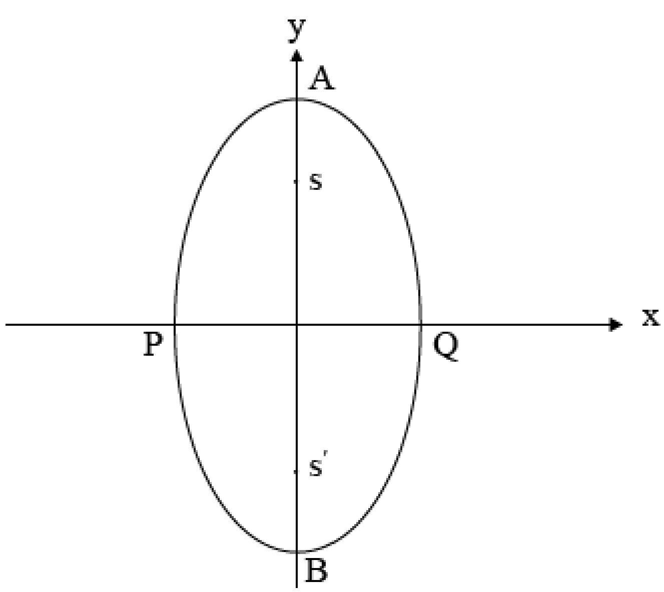
2. MINOR AXIS

Is the one whose length is small

a)



b)



Where

AB – Major axis

PQ – Minor axis

EQUATION OF AN ELLIPSE

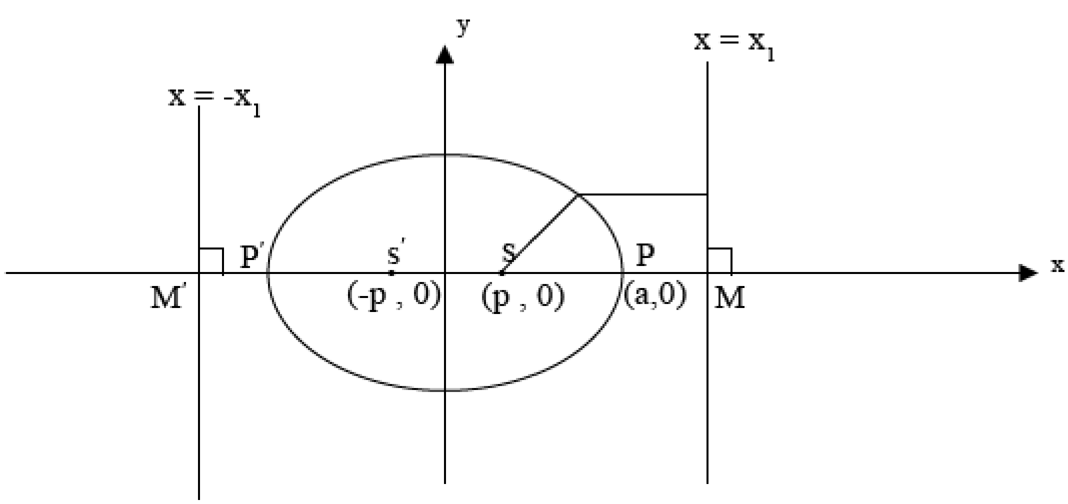
These are;

i) Standard equation

ii) General equation

1. **STANDARD EQUATION**

- Consider an ellipse below;



$$\frac{\overline{SP}}{\overline{MP}} = e, \frac{\overline{SP'}}{\overline{MP'}} = e$$

$$\frac{\overline{SP}}{\overline{MP}} = e$$

$$\overline{SP} = e\overline{MP}$$

$$a - p = e(x_1 - a)$$

$also$

$$\frac{\overline{SP}}{\overline{MP}} = e$$

$$\overline{SP}' = e\overline{MP}'$$

$$(P — a) = e(x_1 — a)$$

$$p + a = e(x_1 + a)$$

$solving\ i\ and\ ii\ as\ follows$

$$+\begin{cases} a - p = e(x_1 — a) \\ a + p = e(x_1 + a) \end{cases}$$

$$2a = e(x_1 — a) + e(x_1 + a)$$

$$2a = e(x_1 — a + x_1 + a)$$

$$2a = e(2x_1)$$

$$2a = 2ex_1$$

$$a = ex_1$$

$$x_1 = \frac{a}{e}$$

$also\ from$

$$a - p = e(x_1 - a)$$

$$a - p = e\left(\frac{a}{e} - a\right)$$

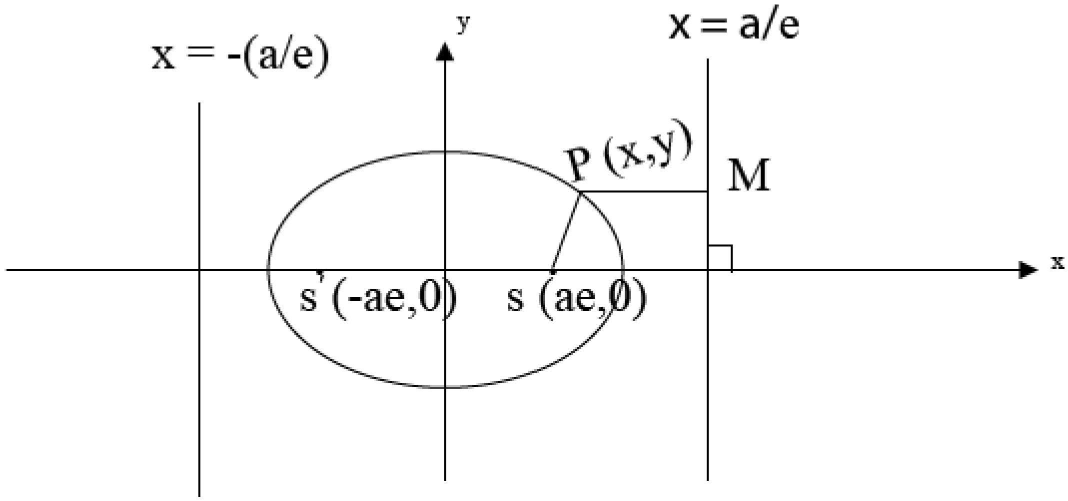
$a - p = a - ae$

$-p = -ae$

$p = ae$

1st CASE

Consider an ellipse along x – axis



$$\frac{\overline{SP}}{\overline{MP}} = e$$

$\overline{SP} = \boldsymbol{e}\overline{MP}$

SQUARING BOTH SIDES

$(\overline{SP})^2 = (\boldsymbol{e}\overline{MP})^2$

$\overline{SP}^2 = e^2(\overline{MP})^2$

$$(x - ae)^2 + (y - 0)^2 = e^2\left[\left(x - \frac{a}{e}\right)^2 + (y - y)^2\right]$$

$$(x - ae)^2 + y^2 = e^2\left[\left(x - \frac{a}{e}\right)^2\right]$$

$$(x - ae)^2 + y^2 = (ex - a)^2$$

$$x^2 - aex - aex + a^2e^2 + y^2 = e^2x^2 - aex - aex + a^2$$

$$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - aex - aex + a^2$$

$$x^2 + a^2e^2 + y^2 = e^2x^2 + a^2$$

$$(x^2 - x^2e^2) + y^2 = (a^2 - a^2e^2)$$

$$\frac{(1 - e^2)x^2}{a^2(1 - e^2)} + \frac{y^2}{a^2(1 - e^2)} = \frac{a^2(1 - e^2)}{a^2(1 - e^2)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

$$put\ a^2(1 - e^2) = b^2$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$where\ a > b$$

$$is\ the\ standard\ equation\ of\ an\ ellipse\ along\ x - axis$$

PROPERTIES

I) an ellipse lies along the x – axis (major axis)

ii) a > b

iii) $a^2(1 - e^2) = b^2$

iv) Foci, $(ae, 0), (-ae, 0)$

v) Directrix $x = \frac{a}{e}, x = -\frac{a}{e}$

vi) Vertices, (a, o), (-a, o) along major axis

(0, b) (0, -b) along minor axis

vi) The length of the major axis l major = 2a

viii) Length of minor axis l minor = 2b

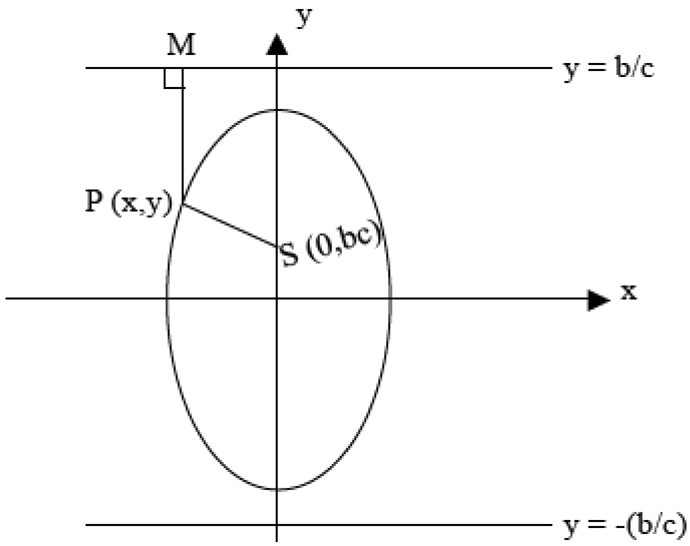
Note:

For an ellipse (a – b) the length along x – axis

B – is the length along y – axis

<u>2nd CASE</u>

- Consider an ellipse along y – axis



$$\frac{\overrightarrow{SP}}{\overrightarrow{MP}} = e$$

$$\overrightarrow{SP} = e\overrightarrow{MP}$$

squaring both sides

$$\left(\overrightarrow{SP}\right)^2 = \left(e\overrightarrow{MP}\right)^2$$

$$\left(\overrightarrow{SP}\right)^2 = e^2\left(\overrightarrow{MP}\right)^2$$

$$(x-0)^2 + (y-be)^2 = e^2\left[(x-x)^2 + \left(y-\frac{b}{e}\right)^2\right]$$

$$x^2 + (y-be)^2 = e^2\left(y-\frac{b}{e}\right)^2$$

$$x^2 + (y-be)^2 = (ey-b)^2$$

$$x^2 + y^2 - bey - bey + b^2e^2 = e^2y^2 - bey - bey + b^2$$

$$x^2 + y^2 + b^2e^2 = e^2y^2 + b^2$$

$$x^2 + y^2 - e^2y^2 = b^2 - b^2e^2$$

$$x^2 + (1 - e^2)y^2 = b^2(1 - e^2)$$

$$\frac{x^2}{b^2(1-e^2)} + \frac{(1-e^2)y^2}{b^2(1-e^2)} = \frac{b^2(1-e^2)}{b^2(1-e^2)}$$

$$\frac{x^2}{b^2(1-e^2)} + \frac{y^2}{b^2} = 1$$

$$but\ b^2(1 - e^2) = a^2$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$where\ b > a$$

$$is\ the\ standard\ equation\ of\ an\ ellipse\ along\ y - axis$$

PROPERTIES

i) An ellipse lies along y – axis (major axis)

ii) b > a

iii) $b^2(1 - e^2) = a^2$

iv) Foci $(0, \pm be)$

v) Directrices $y = \pm\frac{b}{e}$

vi) Vertices: $(0, \pm b)$ = along major

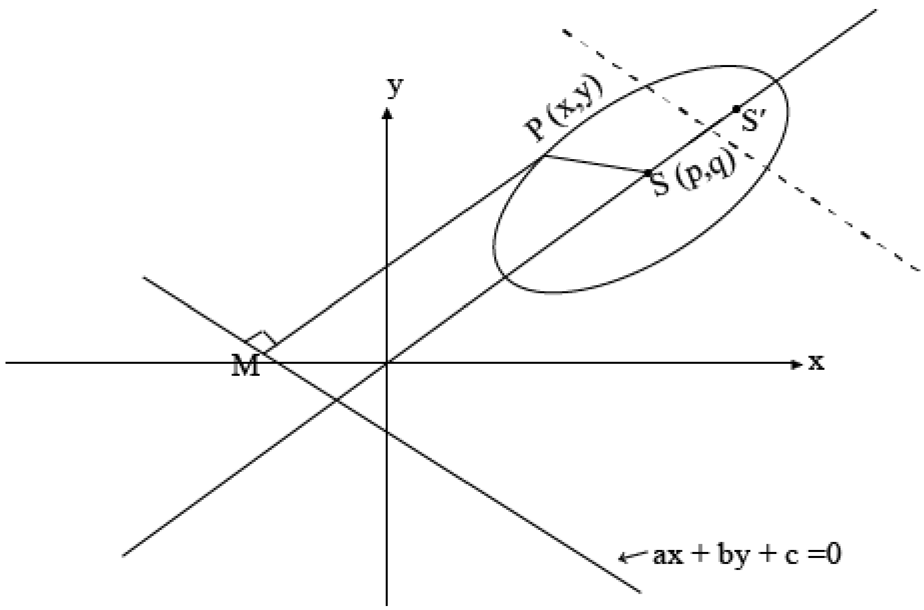
$(\pm a, 0)$ = along minor as

vii) Length of the major axis L major = 2b

viii) Length of the minor axis L minor = 2a

II. GENERAL EQUATION OF AN ELLIPSE

- Consider an ellipse below y – axis



From

$$\frac{\overrightarrow{SP}}{\overrightarrow{MP}} = e$$

$$\sqrt{(x-p)^2 + (y-q)^2} = e\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|$$

$$(x-p)^2 + (y-q)^2 = e^2\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|^2$$

$$(x-p)^2 + (y-q)^2 = e^2\frac{(ax+by+c)^2}{a^2+b^2} \textit{ is the general equation of an ellipse}$$

where

(p, q) *is the focus*

e *is the eccentricity*

EXAMPLE

Given the equation of an ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Find i) eccentricity

ii) Focus

iii) Directrices

Solution

Given $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Compare from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$a^2 = 9 \qquad b^2 = 4$

$a = \pm 3 \qquad b = \pm 2$

$\rightarrow eccentricity$

$from\ a^2(1 - e^2) = b^2$

$a^2(1 - e^2) = 4$

$$1 - e^2 = \frac{4}{9}$$

$$e^2 = 1 - \frac{4}{9}$$

$$e^2 = \frac{5}{9}$$

$$e = \pm\sqrt{\frac{5}{9}}$$

$$e = \pm\frac{\sqrt{5}}{3}$$

$\rightarrow foci$

$foci = (\pm ae, 0)$

$$= \left(\pm 3\left(\frac{\sqrt{5}}{3}, 0\right)\right)$$

$= (\pm\sqrt{5}, 0)$

$\rightarrow directrix$

$$x = \pm\frac{a}{e}$$

$$x = \pm\frac{3}{\frac{\sqrt{5}}{3}}$$

$$\therefore x = \pm\frac{9\sqrt{5}}{5}$$

Find the focus and directrix of an ellipse $9x^2 + 4y^2 = 36$

Solution

Given;

$9x^2 + 4y^2 = 36$

$$\frac{9x^2}{36} + \frac{4y^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

comparing from

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$a^2=4 \qquad b^2=9$

$a=\pm 2 \qquad b=\pm 3$

$\rightarrow b^2(1-e^2)=a^2$

$9(1-e^2)=4$

$$1-e^2=\frac{4}{9}e^2=1-\frac{4}{9}$$

$$e=\pm\sqrt{\frac{5}{9}}$$

$$e=\pm\frac{\sqrt{5}}{3}$$

$\rightarrow foci=(0,\pm be$

$$=\left(0,\pm 3\frac{\sqrt{5}}{3}\right)$$

$$=(0,\pm\sqrt{5})$$

$\rightarrow directrix$

$$y=\pm\frac{b}{e}$$

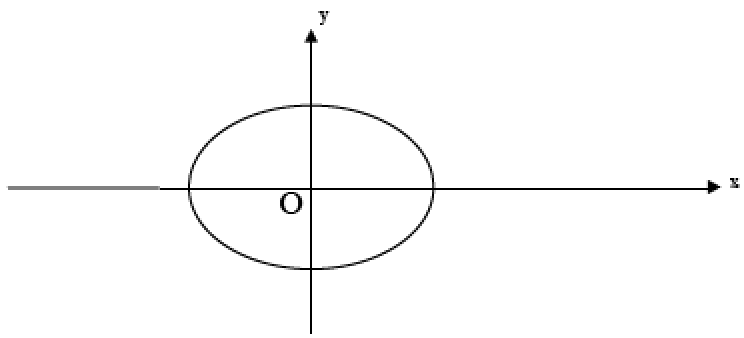
$$y=\pm\frac{3}{\frac{\sqrt{5}}{3}}$$

$$y = \pm\frac{9}{\sqrt{5}}$$

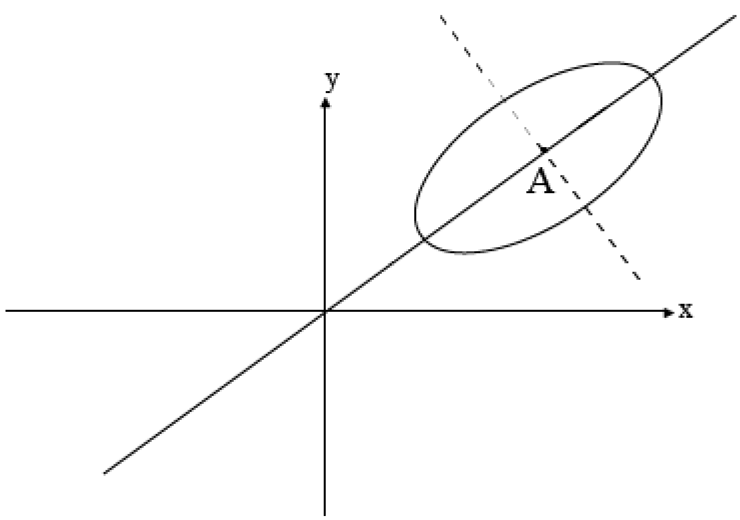
$$\therefore y = \pm\frac{9\sqrt{5}}{5}$$

CENTRE OF AN ELLIPSE

This is the *point of intersection* between major and minor axes



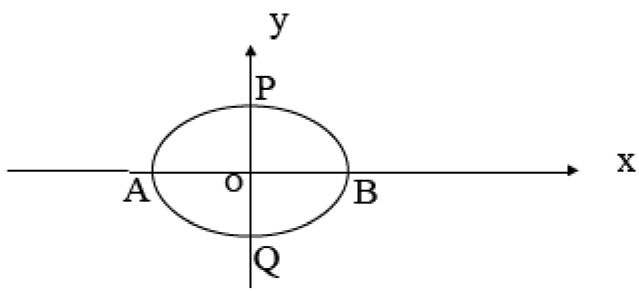
- O – Is the centre of an ellipse



A – Is the centre of an ellipse

DIAMETER OF AN ELLIPSE.

This is any chord passing through the centre of an ellipse



Hence $\overrightarrow{AB}$ – diameter (major)

$\overrightarrow{PQ}$ – Diameter (minor)

Note:

i) The equation of an ellipse is in the form of

$ax^2 + by^2 + cx + dy + exy + f = 0$

$e \rightarrow is\ not\ eccentricity\ \ is\ a\ constant\ variable$

ii) The equation of the parabola is in the form of

$ax^2 + ay^2 + bx + cy + dxy + e = 0$

iii) The equation of the circle is in the form of

$ax^2 + ay^2 + bx + cy + d = 0$

PARAMETRIC EQUATIONS OF AN ELLIPSE

The parametric equations of an ellipse are given as

$x = a\cos\theta$ And $y = b\sin\theta$

Where

θ – Is an eccentric angle

Recall

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

$$put\ \frac{x}{a}=\cos\theta\,,\frac{y}{b}=\sin\theta$$

$$(\cos\theta)^2+(\sin\theta)^2=1$$

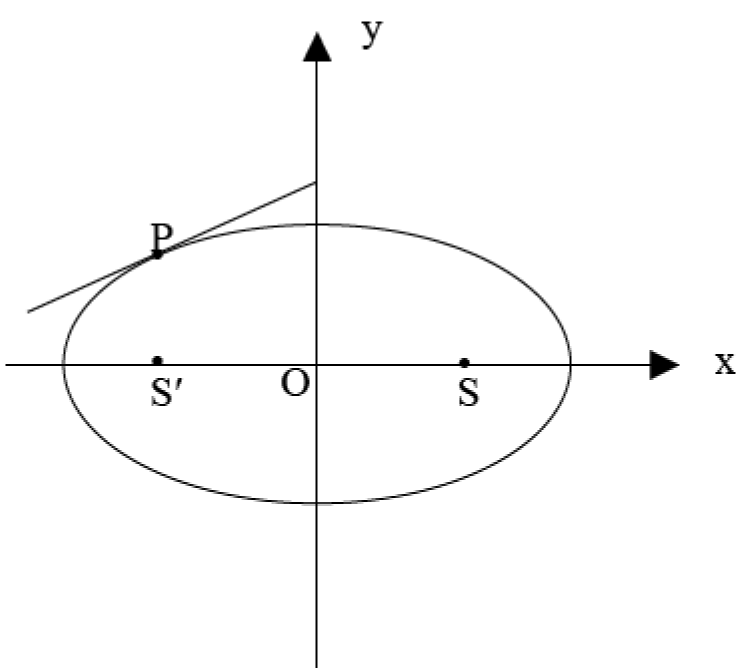
$$cos^2\theta+sin^2\theta$$

$$therefore$$

$$x=a\cos\theta\,,y=b\sin\theta$$

TANGENT TO AN ELLIPSE

This is the straight line which touches the ellipse at only one point



Where;

P – Is the point of tangent or contact

Condition for tangency to an ellipse

Consider the line b = mx + c is the tangent to an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

the condition for tangency is obtained as follows

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

$$b^2x^2+a^2(mx+c)^2=a^2b^2$$

$$b^2x^2+a^2(m^2x^2+2mcx+c^2)=a^2b^2$$

$$b^2x^2+a^2m^2x^2+2mca^2x+a^2c^2-a^2b^2=0$$

$$(b^2+a^2m^2)x^2+(2mca^2)x+(a^2c^2-a^2b^2)=0$$

generally

$$x=\frac{-2mca^2\pm\sqrt{(2mca^2)^2-4(b^2+a^2m^2)(a^2c^2-a^2b^2)}}{2(b^2+a^2m^2)}$$

but discriminant $=0$

$$(2mca^2)^2-4(b^2+a^2m^2)(a^2c^2-a^2b^2)=0$$

$$4m^2c^2a^4=4a^2(b^2+a^2m^2)(c^2-b^2)$$

$$m^2c^2a^2=(b^2+a^2m^2)(c^2-b^2)$$

$$m^2c^2a^2=b^2c^2-b^4+m^2c^2a^2-a^2m^2b^2$$

$$0=b^2c^2-b^4-a^2m^2b^2$$

$$b^4+a^2m^2b^2=b^2c^2$$

$$c^2=a^2m^2+b^2$$

Examples

Show that, for a line $y=mx+c$ to touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Then,

$$c^2=a^2m^2+b^2$$

GRADIENT OF TANGENT TO AN ELLIPSE

This can be expressed into;

i) Cartesian form

ii) Parametric form

1. GRADIENT OF TANGENT IN CARTESIAN FORM

- Consider an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiate both sides with w.r.t x

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{2y}{b^2}\frac{dy}{dx}=-\frac{2x}{a^2}$$

$$\frac{dy}{dx}=\frac{-2x.b^2}{2ya^2}$$

$$\frac{dy}{dx}=\frac{-b^2x}{a^2y}=m$$

$$\therefore m=\frac{-b^2x}{a^2y}$$

ii. GRADIENT OF TANGENT IN PARAMETRIC FORM

- Consider the parametric equation of an ellipse

$$x=a\cos\theta\,,y=b\sin\theta$$

$$\rightarrow x=a\cos\theta$$

$$\frac{dx}{d\theta}=-a\sin\theta$$

$$\rightarrow y=b\sin\theta\,\frac{dy}{d\theta}=b\cos\theta$$

$$from$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$$

$$\frac{dy}{dx}=\frac{b\cos\theta}{-a\sin\theta}$$

$$\frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

$$let\ m = \frac{dy}{dx} = slope\ of\ tangent\ in\ parametric\ form$$

$$\therefore m = -\frac{b}{a}\cot\theta$$

EQUATION OF TANGENT TO AN ELLIPSE

These can be expressed into;

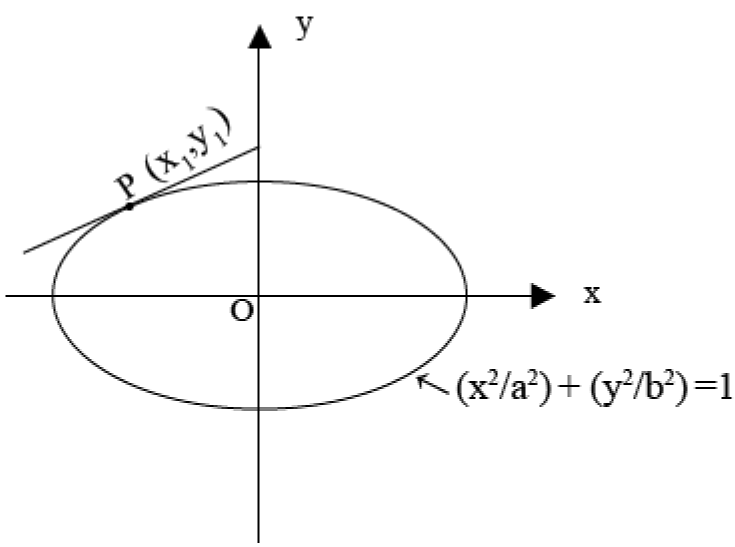
i) Cartesian form

ii) Parametric form

I. Equation of tangent in Cartesian form

- Consider the tangent an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ at\ the\ point\ p(x_1, y_1)$$



Hence, the equation of tangent is given by

$$m = \frac{-b^2 x_1}{a^2 y_1} = \frac{y - y_1}{x - x_1}$$

$$\frac{-b^2 x_1}{a^2 y_1} = \frac{y - y_1}{x - x_1}$$

$$-b^2 x_1 (x - x_1) = a^2 y_1 (y - y_1)$$

$$-b^2 x\ _1 x + b^2 x_1{}^2 = a_2 y_1 y - a_2 y^2{}_1$$

$$b^2 x_1{}^2 + a_2 y^2{}_1 = b^2 x\ _1 x + a_2 y_1 y$$

$divide\ by\ a^2 b^2\ both\ sides$

$$\frac{b^2 x_1{}^2}{a^2 b^2} + \frac{a^2 y_1{}^2}{a^2 b^2} = \frac{b^2 x_1 x}{a^2 b^2} + \frac{a^2 y_1 y}{a^2 b^2}$$

$$\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} = \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2}$$

$but\ from$

$$\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} = \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2}$$

$$\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} = 1$$

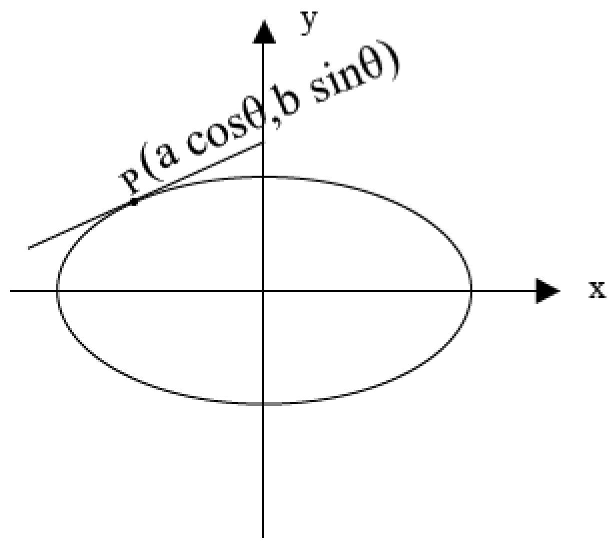
$$1 = \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2}$$

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$$

$is\ the\ equation\ of\ tangent\ in\ cartesian\ form$

EQUATION OF TANGENT IN PARAMETRIC FORM.

Consider the tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ At the point $p(a\cos\theta\,, b\sin\theta)$



Hence the equation of tangent is given by

$$m=-\frac{b}{a}\cot\theta=\frac{y-b\sin\theta}{x-a\cos\theta}$$

$$-\frac{b}{a}\frac{\cos\theta}{\sin\theta}=\frac{y-b\sin\theta}{x-a\cos\theta}$$

$$-b\cos\theta\,(x-a\cos\theta)=a\sin\theta(y-b\sin\theta)$$

$$-bx\cos\theta+abcos^2\theta=ay\sin\theta-ab\ sin^2\theta$$

$$-bx\cos\theta-ay\sin\theta+sab(cos^2\theta+sin^2\theta)=0$$

$$-bx\cos\theta-ay\sin\theta+ab=0$$

$$bx\cos\theta+ay\sin\theta-ab=0$$

$$bx\cos\theta+ay\sin\theta=ab$$

$dividing\ \ bu\ ab\ both\ sides$

$$\frac{bx\cos\theta}{ab}+\frac{y\sin\theta}{b}=1$$

$$\therefore \frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

$is\ the\ equation\ of\ tangent\ in\ parametric\ form$

Note

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$x\frac{(x)}{a^2}+y\frac{(y)}{b^2}=1$$

$$put\ x=x_1, y=y_1$$

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$$

2. $\frac{x^2}{b^2}+\frac{y^2}{b^2}=1$

$$\frac{x}{a}\left(\frac{x}{a}\right)+\frac{y}{b}\left(\frac{y}{b}\right)=1$$

$$put\ \frac{x}{a}=\cos\theta$$

$$\frac{y}{b}=\sin\theta$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta$$

EXERCISE

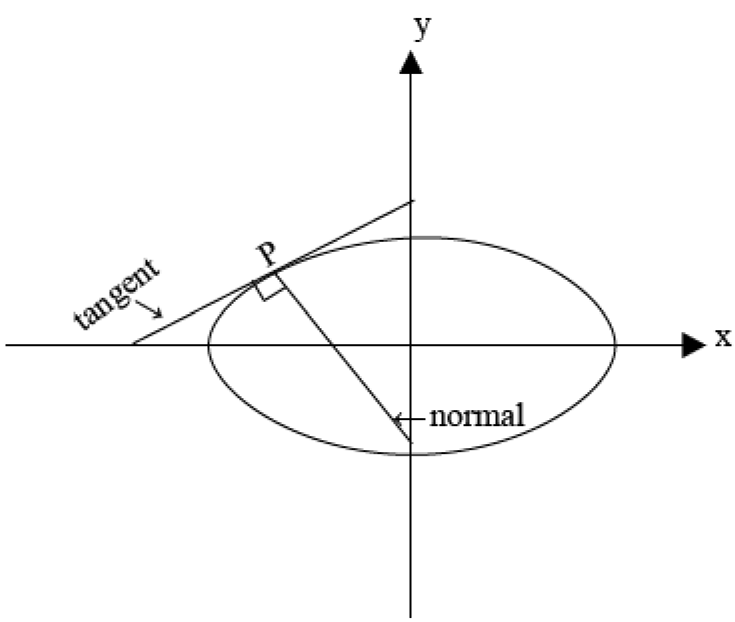
i. Show that the equation of tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *at the point* (x_1, y_1) *is* $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$

ii. Show that the equation of tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *at* $(a\cos\theta, b\sin\theta)$ *is* $bx\cos\theta + ay\sin\theta - ab = 0$

iii. Show that the gradient of tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is* $m = \frac{-b^2 x}{a^2 y}$ *hence state for* $a > b$ *and* $b > a$

NORMAL TO AN ELLIPSE

Normal to an ellipse perpendicular to the tangent at the point of tangency.



Where: p is the point of tangency

GRADIENT OF THE NORMAL TO AN ELLIPSE.

This can be expressed into two

i) Cartesian form
ii) Parametric

I) IN CARTESIAN FORM

- Consider the gradient of the tangent in Cartesian form

But normal tangent

$$m_1 = \frac{-b^2x}{a^2y}$$

let m be the gradient of the normal in cartesian form

but normal is perpendicular to the tangent

$$m_1m_2 = -1$$

$$m = \frac{-1}{m_1}$$

$$m = \frac{-1}{\frac{-b^2x}{a^2y}}$$

$$m = \frac{a^2y}{b^2x}$$

II) <u>IN PARAMETRIC FORM</u>

Consider the gradient of tangent in parametric form

$$m_1 = -\frac{b}{a}\cot\theta$$

Let m = slope of the normal in parametric form

$$m_1m_2 = -1$$

$$m = \frac{-1}{-\frac{b}{a}\cot\theta}$$

$$m = \frac{a}{b}\tan\theta$$

<u>EQUATION OF THE NORMAL TO AN ELLIPSE</u>

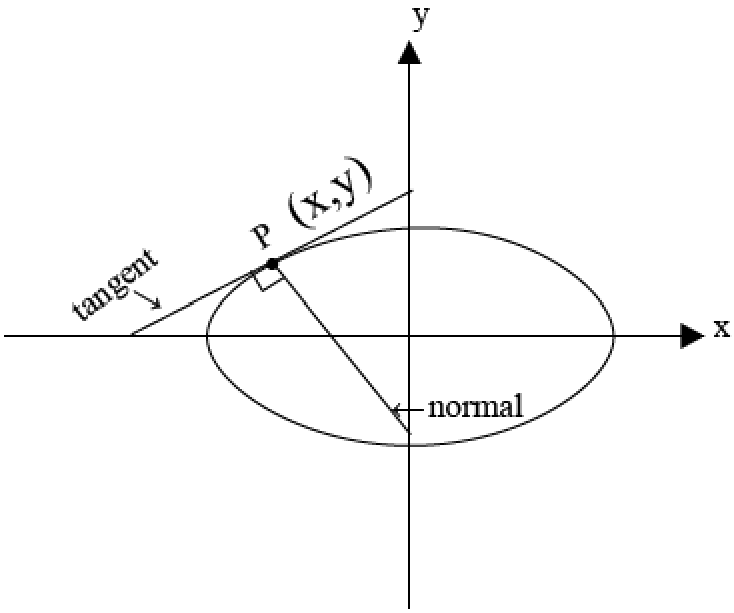
This can be expressed into;

(i) Cartesian form

(ii) Parametric form

I. <u>IN CARTESIAN FORM</u>

- consider the normal to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ at\ point\ p(x,y_1)$



Hence the equation of the normal is given by

$$m=\frac{a^2y_1}{b^2x_1}=\frac{y-y_1}{x-x_1}$$

$$\frac{a^2y_1}{b^2x_1}=\frac{y-y_1}{x-x_1}$$

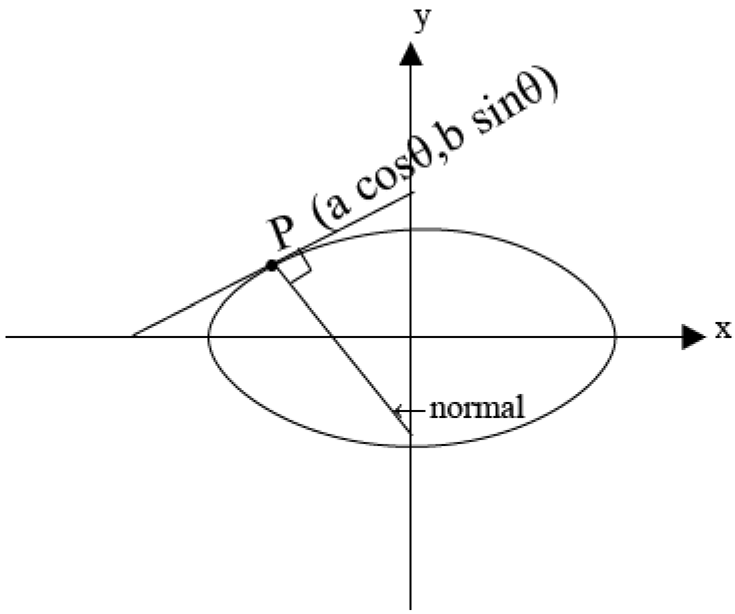
$$b^2x_1(y-y_1)=a^2y_1(x-x_1)$$

$$b^2x_1y-b^2x_1y_1=a^2y_1x-a^2x_1y_1$$

$$\rightarrow a^2y_1x-b^2x_1y+x_1y_1(b^2-a^2)=0\ is\ the\ equation\ of\ normal\ in\ cartesian\ form$$

II) IN PARAMETRIC FORM

Consider the normal to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $at\ point\ p(a\cos\theta, b\sin\theta)$



Hence the equation of the normal is given by

$$m=\frac{a}{b}\tan\theta=\frac{y-b\sin\theta}{x-a\cos\theta}$$

$$\frac{a}{b}\tan\theta=\frac{y-b\sin\theta}{x-a\cos\theta}$$

$$\frac{a\sin\theta}{b\cos\theta}=\frac{y-b\sin\theta}{x-a\cos\theta}$$

$$a\sin\theta\,(x-a\cos\theta)=b\cos\theta\,(y-b\sin\theta)$$

$$ax\sin\theta-a^2\sin\theta\cos\theta=by\cos\theta-b^2\sin\theta\cos\theta$$

$$ax\sin\theta-by\cos\theta-a^2\sin\theta\cos\theta+b^2\sin\theta\cos\theta=0$$

$$ax\sin\theta-by\cos\theta-(a^2-b^2)\sin\theta\cos\theta=0$$

$$\therefore ax\sin\theta-by\cos\theta+\frac{1}{2}(b^2-a^2)\sin 2\theta=0$$

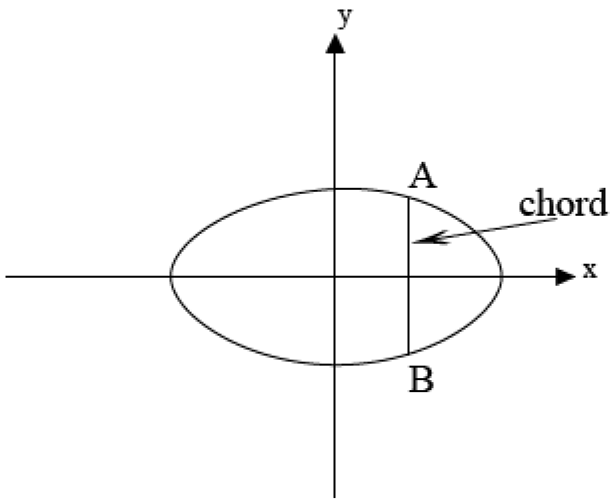
is the equation of the normal in parametric form

Examples

- Show that the equation of the normal to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ at(x_1,y_1) is\ a^2x_1y=x_1y_1(a^2-b^2)$

CHORD OF AN ELLIPSE.

This is the line joining any two points on the curve ie (ellipse)



GRADIENT OF THE CHORD TO AN ELLIPSE.

This can be expressed into

i) Cartesian form

ii) Parametric form

I. IN CARTESIAN FORM

- Consider the point A (x_1, y_1) and B (x_2, y_2) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ hence the gradient of the cord is given by $m, = \frac{y_2 - y_1}{x_2 - x_1}$

II. IN PARAMETRIC FORM

Consider the points A $(a\cos\theta\,, b\sin\theta)$ and B $(a\cos\emptyset, b\sin\emptyset)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Hence the gradient of the chord is given by;

$$m = \frac{b\sin\theta - b\sin\emptyset}{a\cos\theta - a\cos\emptyset}$$

$$m = \frac{b}{a}\left[\frac{\sin\theta - \sin\emptyset}{\cos\theta - \cos\emptyset}\right]$$

$$m = \frac{b}{a}\left[\frac{2\cos\left(\frac{\theta+\emptyset}{2}\right)\sin\left(\frac{\theta+\emptyset}{2}\right)}{-2\left[\sin\left(\frac{\theta+\emptyset}{2}\right)\cos\left(\frac{\theta+\emptyset}{2}\right)\right]}\right]$$

$$\therefore m = -\frac{b}{a}\cot\left(\frac{\theta+\emptyset}{2}\right)$$

EQUATION OF THE CHORD TO AN ELLIPSE

These can be expressed into
i) Cartesian form
ii) Parametric form

I: IN CARTESIAN FORM.

Consider the chord the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point A (x_1, y_1) and B(x_2,y_2). Hence the equation of the chord is given by;

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{y-y_1}{x-x_1}$$

$$\frac{y_2-y_1}{x_2-x_1} = \frac{y-y_1}{x-x_1}$$

$$(y_2-y_1)(x-x_1) = (y-y_1)(x_2-x_1)$$

$$= y_2x - y_2x_1 - y_1x + y_1x_1 = yx_2 - yx_{1-y_1x_2} + y_1x_1$$

$$\therefore (y_2-y_1)x - (x_2-x_1)y + (y_1x_2 - y_2x_1) = 0$$

$is\ the\ equation\ of\ the\ chord\ in\ cartesian\ form$

II. IN PARAMETRIC FORM.

Consider the chord to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points $(a\cos\theta, b\sin\emptyset)$. Hence the equation of the chord is given by

$$m = -\frac{b}{a}\cot\left(\frac{\theta+\emptyset}{2}\right) = \frac{y-b\sin\theta}{x-a\cos\theta}$$

$$-\frac{b}{a}\cot\left(\frac{\theta+\emptyset}{2}\right) = \frac{y-b\sin\theta}{x-a\cos\theta}$$

$$\frac{-b\cos\left(\frac{\theta+\emptyset}{2}\right)}{a\sin\left(\frac{\theta+\emptyset}{2}\right)}=\frac{y-b\sin\theta}{x-a\cos\theta}$$

$$b\cos\left(\frac{\theta+\emptyset}{2}\right)(x-a\cos\theta)=a\sin\left(\frac{\theta+\emptyset}{2}\right)(y-b\sin\theta)$$

$$-bx\cos\left(\frac{\theta+\emptyset}{2}\right)+ab\cos\left(\frac{\theta+\emptyset}{2}\right)\cos\theta=ay\sin\left(\frac{\theta+\emptyset}{2}\right)-ab\sin\left(\frac{\theta+\emptyset}{2}\right)\sin\theta$$

$$-bx\cos\left(\frac{\theta+\emptyset}{2}\right)-ay\sin\left(\frac{\theta+\emptyset}{2}\right)+ab\left[\cos\left(\frac{\theta+\emptyset}{2}\right)\cos\theta+\sin\left(\frac{\theta+\emptyset}{2}\right)\sin\theta\right]=0$$

$$-bx\cos\left(\frac{\theta+\emptyset}{2}\right)-ay\sin\left(\frac{\theta+\emptyset}{2}\right)+ab\,cos\left[\frac{\theta+\emptyset}{2}-\theta\right]=0$$

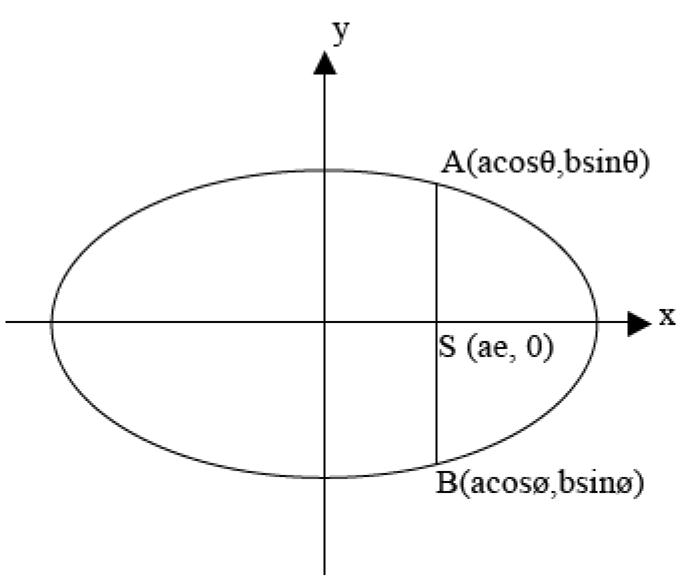
$$-bx\cos\left(\frac{\theta+\emptyset}{2}\right)-ay\sin\left(\frac{\theta+\emptyset}{2}\right)+ab\cos\left(\frac{\emptyset-\theta}{2}\right)=0$$

$$\therefore\ bx\cos\frac{1}{2}(\emptyset+\theta)+ay\sin\frac{1}{2}(\emptyset+\theta)-ab\cos\frac{1}{2}(\emptyset-\theta)=0$$

$is\ the\ equationnof\ the\ chord\ in\ parametric\ form$

FOCAL CHORD OF AN ELLIPSE.

This is the chord passing through the focus of an ellipse



Where $\overline{ASD}$ = is the focal chord

Consider the points A and B are respectively $(a\cos\theta, b\sin\theta) and\ (a\cos\emptyset, b\sin\emptyset)$ Hence

Gradient of AS = gradient of BS

Where s = (ae, o)

$$\frac{b\sin\theta - 0}{a\cos\theta - ae} = \frac{b\sin\emptyset - 0}{a\cos\emptyset - ae}$$

$$\frac{b\sin\theta}{a\cos\theta - ae} = \frac{b\sin\emptyset}{a\cos\emptyset - ae}$$

$$\frac{b}{a}\left[\frac{\sin\theta}{\cos\theta - e}\right] = \frac{b}{a}\left[\frac{\sin\emptyset}{\cos\emptyset - e}\right]$$

$$\frac{\sin\theta}{\cos\theta - e} = \frac{\sin\emptyset}{\cos\emptyset - e}$$

$$\sin\theta(\cos\emptyset - e) = \sin\emptyset(\cos\theta - e)$$

$$\sin\theta\cos\emptyset - e\sin\theta = \sin\emptyset\cos\theta - e\sin\emptyset$$

$$\sin\theta\cos\emptyset - \cos\theta\sin\emptyset = e\sin\theta - e\sin\emptyset$$

$$\sin(\theta - \emptyset) = e(\sin\theta - \sin\emptyset)$$

$$\sin(\theta - \emptyset) = e\left[2\cos\left(\frac{\theta + \emptyset}{2}\right)\sin\left(\frac{\theta - \emptyset}{2}\right)\right]$$

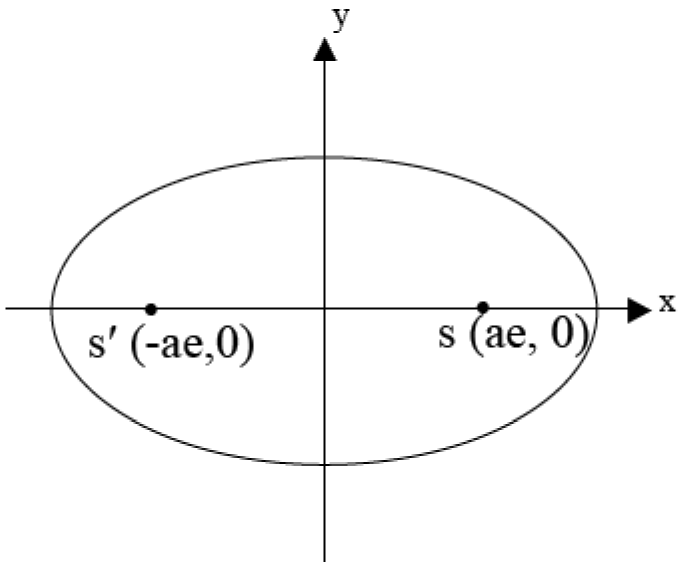
$$2\sin\left(\frac{\theta - \emptyset}{2}\right)\cos\left(\frac{\theta - \emptyset}{2}\right) = 2e\cos\left(\frac{\theta + \emptyset}{2}\right)\sin\left(\frac{\theta - \emptyset}{2}\right)$$

$$\cos\left(\frac{\theta - \emptyset}{2}\right) = e\cos\left(\frac{\theta + \emptyset}{2}\right)$$

$$\therefore \mathbf{\cos\frac{1}{2}(\theta - \emptyset) = e\cos\frac{1}{2}(\theta + \emptyset)}$$

<u>DISTANCE BETWEEN TWO FOCI.</u>

Consider the ellipse below;



$$\overline{SS'} = \sqrt{(ae - ae)^2 + (0 - 0)_2}$$

$$\overline{SS'} = \sqrt{(ae + ae)_2}$$

$$\overline{SS'} = \sqrt{(2ae)_2}$$

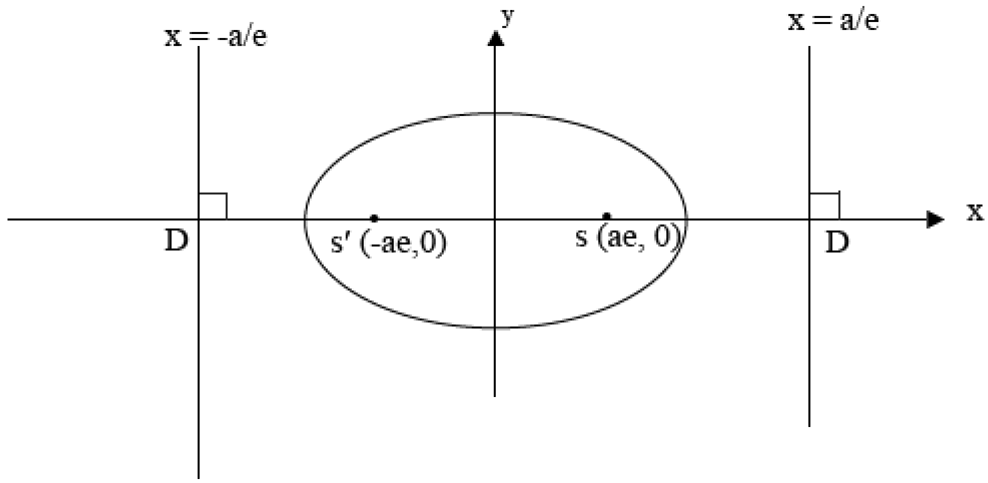
$$\overline{SS'} = 2ae$$

Where a = is the semi major axis

e = is the eccentricity

DISTANCE BETWEEN DIRECTRICES.

Consider the ellipse below



$= \sqrt{\left(\frac{a}{e} - \frac{a}{e}\right)^2 + (0-0)^2}$

$= \sqrt{\left(\frac{a}{e} + \frac{a}{e}\right)^2}$

$= \sqrt{\left(\frac{2a}{e}\right)^2}$

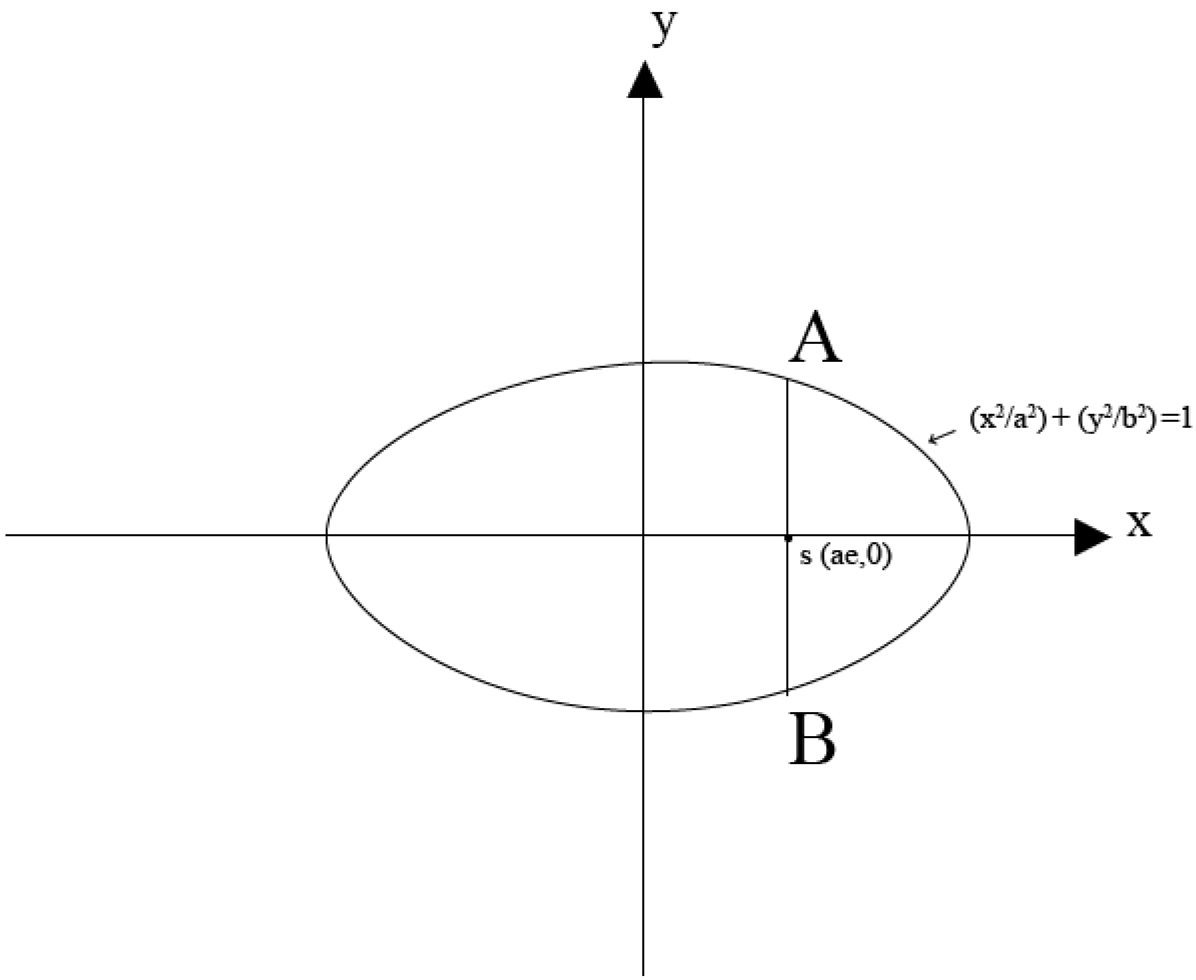
$\overline{DD'} = \frac{2a}{e}$

Where *a* – is the semi major axis

e – is the eccentricity

LENGTH OF LATUS RECTUM.

Consider the ellipse below



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$put\ x=ae$$

$$\frac{(ae)^2}{a^2}+\frac{y^2}{b^2}=1$$

$$e^2+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-e^2$$

$y^2 = b^2(1 - e^2)$

$but\ a^2(1 - e^2) = b^2$

$b^2 = a^2(1 - e^2)$

$y^2 = a^2(1 - e^2)(1 - e^2)$

$y = \pm\sqrt{a^2\left((1 - e^2)\right)^2}$

$y = \pm a(1 - e^2)$

$\therefore\ the\ length\ of\ latus\ rectum\ is\ given\ by$

$L_r = 2a(1 - e^2) also$

$L_r = 2a(1 - e^2)$

$but\ a^2(1 - e^2) = b^2$

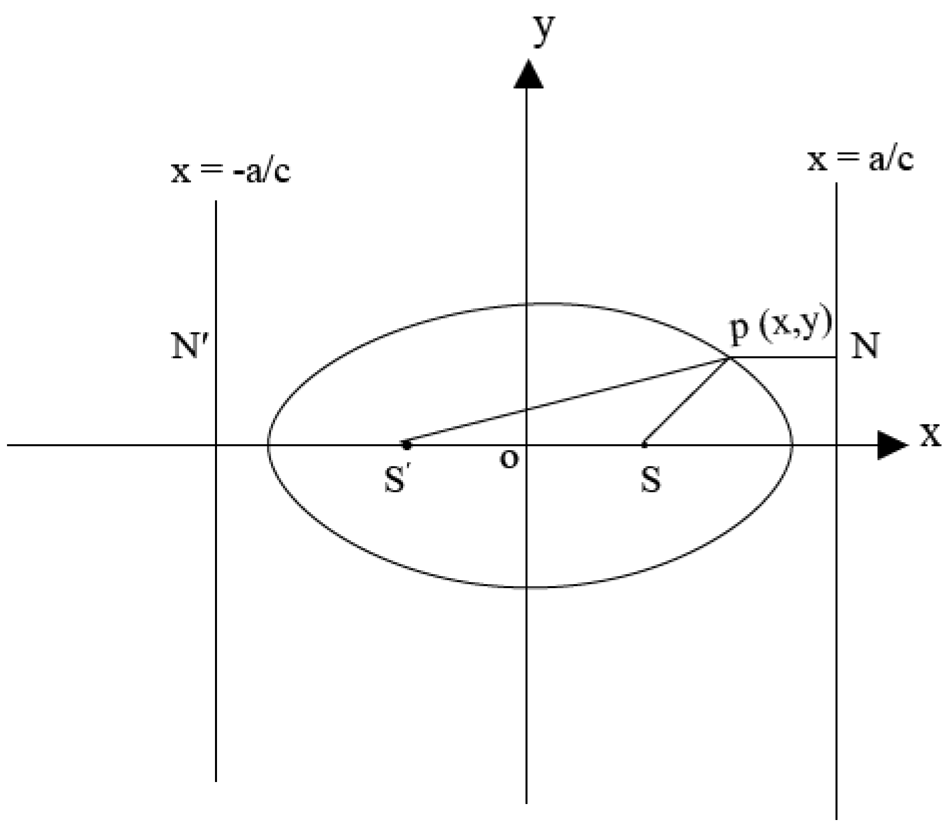
$1 - e^2 = \frac{b^2}{a^2}$

$L_r = 2a\left(\frac{b}{a}\right)^2$

$\therefore L_r = \frac{2b^2}{a}$

<u>IMPORTANT RELATION OF AN ELLIPSE</u>

Consider an ellipse below



$$\frac{\overline{PS}}{\overline{PN}} = e$$

$$\overrightarrow{PS} = e\overrightarrow{PN} \dots\dots\dots (i)$$

$$\frac{\overline{PS'}}{\overline{PN'}} = e$$

$$\overline{PS'} = e\overline{PN'} \dots\dots\dots (ii)$$

adding equation (i) *and* (ii)

$$+\begin{cases}\overrightarrow{PS} = e\overrightarrow{PN} \\ \overline{PS'} = e\overline{PN'}\end{cases}$$

$$\overline{PS} + \overline{PS'} = e\overline{PN} + e\overline{PN'}$$

$$\overline{PS} + \overline{PS'} = e(\overline{PN} + \overline{PN'})$$

$$\overline{PS} + \overline{PS'} = e\left(\frac{2a}{e}\right)$$

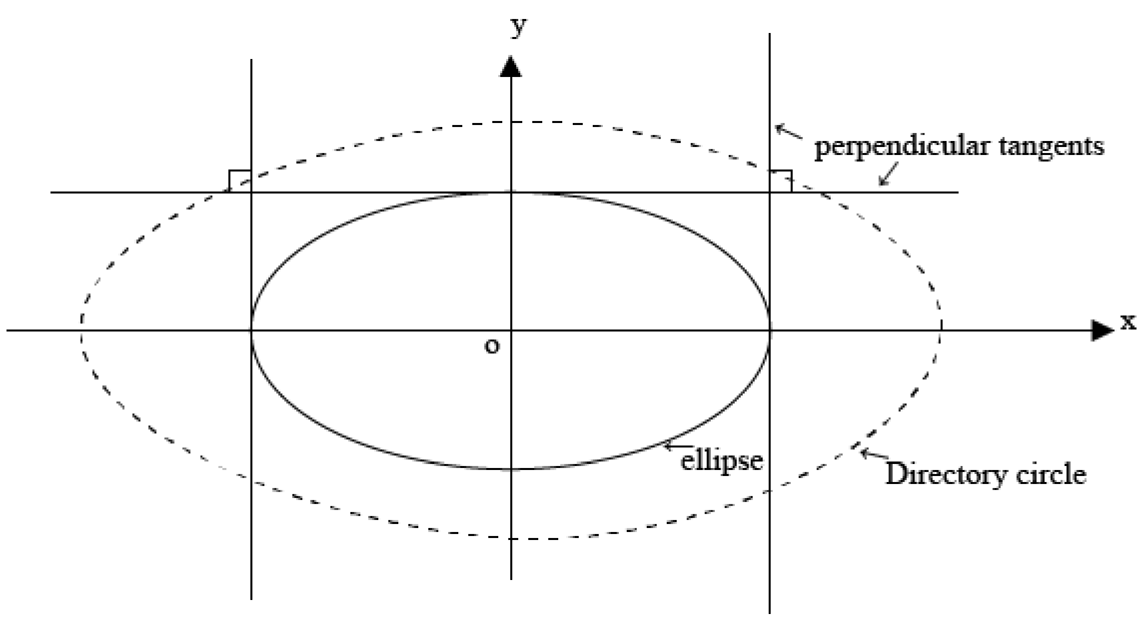
$$\therefore \overline{PS} + \overline{PS'} = 2a$$

$$where\ \overline{PS}\ and\ \overline{PS'}\ are\ the\ focal\ distance\ and\ 2a\ is\ the\ major\ axis$$

ECCENTRIC ANGLE OF ELLIPSE

.This is the angle introduced in the parametric equation of an ellipse
I.e $x = a\cos\theta, y = b\sin\theta$
Where θ - is an eccentric angle
CIRCLES OF AN ELLIPSE
These are 1) Director Circle
2) Auxiliary circle

1. DIRECTOR CIRCLE
- This is the locus of the points of intersection of the perpendicular tangents.



Consider the line $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Hence
$c^2 = a^2m^2 + b^2$

$c = \pm\sqrt{a^2m^2 + b^2}$

$\rightarrow y = mx + c$

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$y - mx = \left(\pm\sqrt{a^2m^2 + b^2}\right)$

$(y - mx)^2 = \left(\pm\sqrt{a^2m^2 + b^2}\right)^2 (y - mx)^2 = a^2m^2 + b^2$

$y^2 - mxy - mxy + m^2x^2 = a^2m^2 + b^2$

$y^2 - 2mxy + m^2x^2 = a^2m^2 + b^2$

$(m^2x^2 - a^2m^2) - 2mxy + (y^2 - b^2) = 0$

$(x^2 - a^2)m^2 + (-2xy)m + y^2 - b^2$

$let\ m_1\ and\ m_2\ be\ the\ gradients. the\ perpendicular\ tangents\ m_1m_2 = -1$

$also\ m_1m_2\ are\ roots\ of\ quadratic\ equation\ above$

$\rightarrow m_1m_2 = product\ of\ roots$

$m_1m_2 = \frac{y^2 - b^2}{x^2 - a^2} = 1$

$y^2 - b^2 = -1(x^2 - a^2)$

$y^2 - b^2 = -x^2 + a^2$

$x^2 + y^2 = a^2 + b^2$

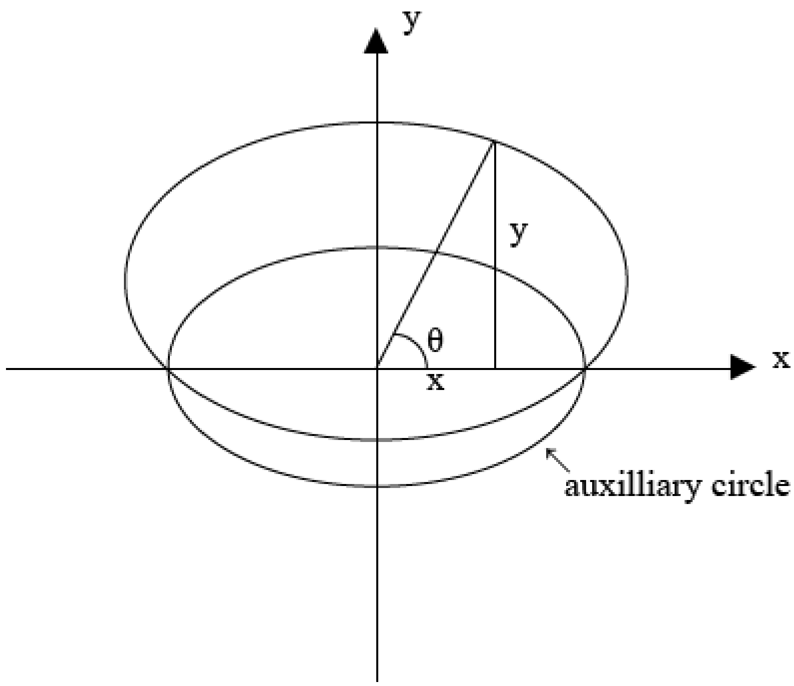
$\rightarrow x^2 + y^2 = a^2 + b^2$

$is\ the\ general\ equation\ of\ a\ director\ circle$

$where\ a^2 + b^2 = r^2\ and\ r \rightarrow is\ the\ radius\ of\ director\ circle$

2. AUXILIARY CIRCLE

- This is the circle whose radius is equal to semi – major axis



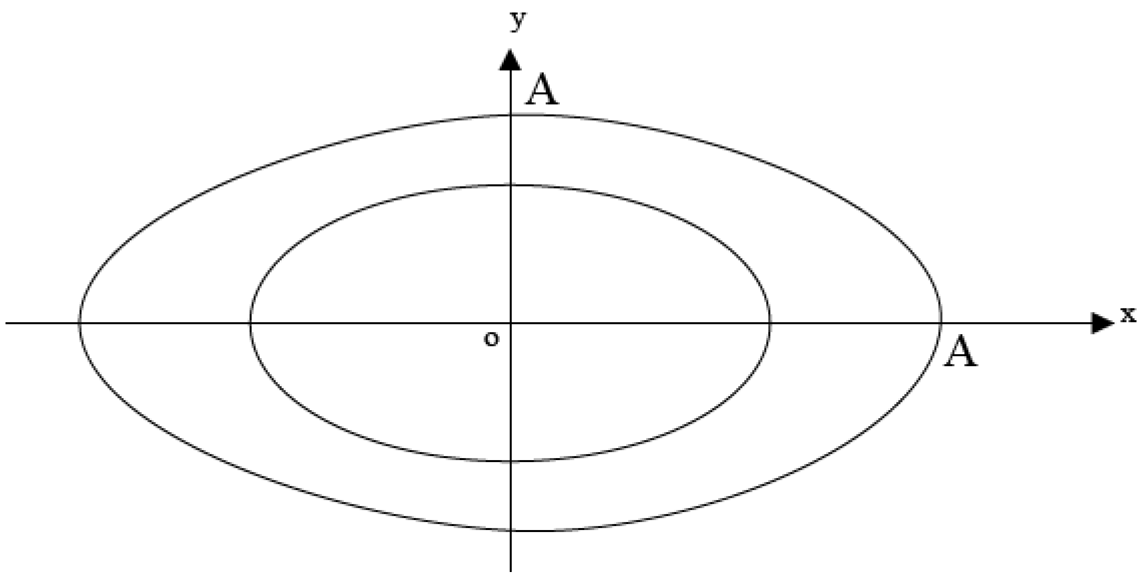
Using Pythagoras theorem

$x^2 + y^2 = a^2$ *is the equation of auxilliary circle*

a – is the radius of the auxiliary circle

CONCENTRIC ELLIPSE.

These are ellipse whose centre are the same.



The equations of centric ellipse are;

$$\frac{y^2 - b^2}{x^2 - a^2} = 1 \; and \; \frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1$$

Where a and b semi – major and semi – minor axes of the small ellipse

A and B are the semi – major and semi – minor axes of the large ellipse

A – a = B – b

A – B = a – b·

Is the condition for concentric ellipse

TRANSLATED ELLIPSE

This is given by the equation

$$\left(\frac{(x-h)}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

A. PROPERTIES

i) An ellipse lies along x – axis
ii) a > b
iii) Centre (h, k)
iv) Vertices $(h \pm a, k)$
v) Eccentricity, $b^2 = a^2(1-e^2)$
vi) Foci $(h \pm ae, k)$
vii) Directrices $x = h \pm \frac{a}{e}$

B. PROPERTIES

i) An ellipse has along y – axis
ii) b > a
iii) Centre (h, k)
iv) Vertices $(h, k \pm b)$
v) Eccentricity $a^2 = b^2(1-e^2)$
vi) Foci $(h, k \pm be)$
vii) Directices $y = k \pm \frac{b}{e}$

Examples

Show that the equation $4x^2 - 16x + 9y^2 + 18y - 11 = 0$ represents an ellipse and hence find i) centre ii) vertices iii) eccentricity iv) foci v) directrices.

<u>Solution</u>

$$4x^2 - 16x + 9y^2 + 18y - 11 = 0$$

$$4(x^2 - 4x) + 9(y^2 + 2y) = 11$$

$$4(x^2 - 4x + 4) + 9(y^2 + 2y + 1) = 11 + 16 + 9$$

$$\frac{4(x-2)^2}{36} + \frac{9(y+1)^2}{36} = \frac{36}{36}$$

$$\frac{(x-1)^2}{9}+\frac{(y+1)^2}{4}=1$$

$put\ x-2=X\ and\ y+1=Y$

$$\frac{X^2}{9}+\frac{Y^2}{4}=1$$

$\therefore the\ equation\ represents\ an\ ellipse$

$$hence\ \frac{(x-2)^2}{9}+\frac{(y+1)^2}{4}=1$$

$$comparing\ from\ \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$h=2, k=-1$

$a^2=9, b^2=4$

$a=\pm 3, b=\pm 2$

$(i) centre=(h,k)$

$(2,-1)$

$(ii) vertices$

$=(\pm a,0)+(h,k)$

$=(\pm 3,0)+(2,+\ -1)$

$\big((2\pm 3),-1\big)$

$=(5,1) and\ (-1,-1)$

$(iii) Eccentricity$

$from\ a^2(1-e^2) = b^2$

$9(1-e^2) = 4$

$1-e^2 = \frac{4}{9}$

$e^2 = 1 - \frac{4}{9}$

$e = \pm\frac{\sqrt{5}}{3}$

$(iv) Foci$

$= (\pm ae, 0) + (h, k)$

$= \left(\pm 3\left(\frac{\sqrt{5}}{3}, 0\right)\right) + (2, -1)$

$= [\pm\sqrt{5}, 0] \pm [2, -1]$

$= (2 \pm \sqrt{5}, -1)$

$= (2+\sqrt{5}, 1) and\ (2-\sqrt{5}, -1)$

$(v) Directrices$

$x = \pm\frac{a}{e} + h$

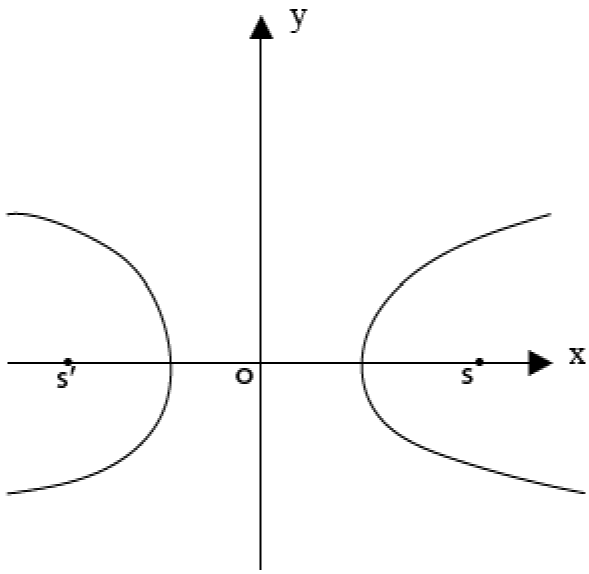
$x = h \pm \frac{a}{e}$

$x = 2 \pm \frac{9}{\sqrt{5}}$

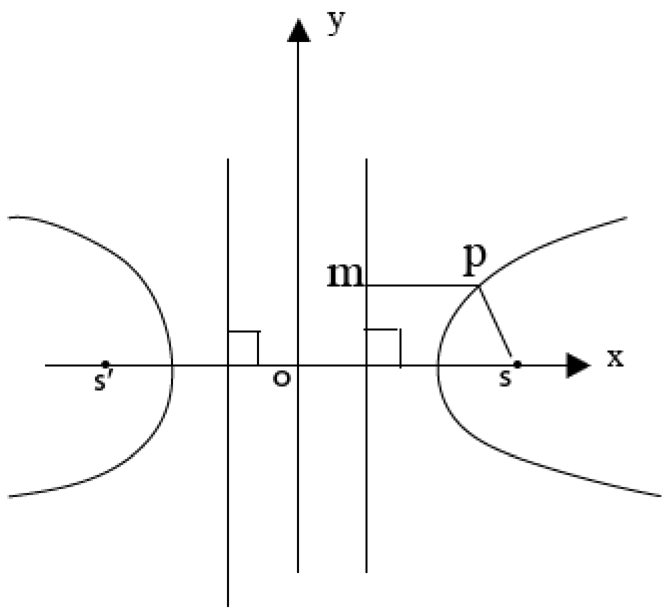
$$x = 2 + \frac{9}{\sqrt{5}} \; and \; x = 2 - \frac{9}{\sqrt{5}}$$

III. <u>HYPERBOLA</u>

This is the conic section whose eccentricity ''e'' is greater than one (e > 1)



The hyperbola has two foci and two directrices



Where S and S' are the foci of the hyperbola hence

$$\frac{\overline{SP}}{\overline{MP}} = e$$

Where e > 1

EQUATION OF THE HYPERBOLA

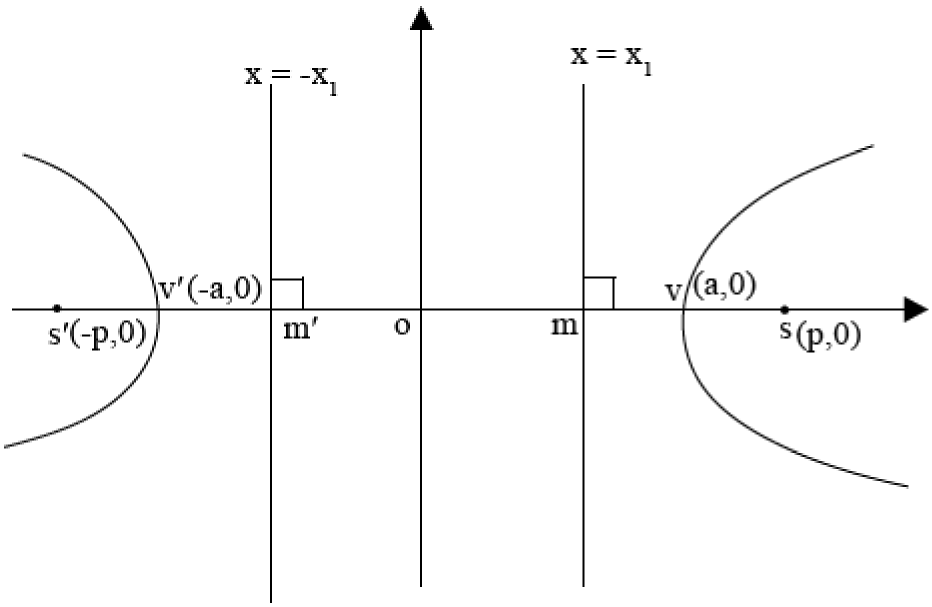
There are;

i) Standard equation

ii) General equation

1. STANDARD EQUATION OF THE HYPERBOLA

Consider



$$\frac{\overline{SV}}{\overline{MV}} = e$$

$$\frac{\overline{SV'}}{\overline{MV'}} = e$$

$$\overline{SV} = e\overline{MV}$$

$$p - a = e(a - x_1) \ldots .(1)$$

$$\rightarrow \frac{\overline{SV'}}{\overline{MV'}} = e$$

$$\overline{SV'} = e\overline{MV}'$$

VECTOR ANALYSIS-1

Definition

Vectors are any quantity that possess both magnitude and direction .

Example

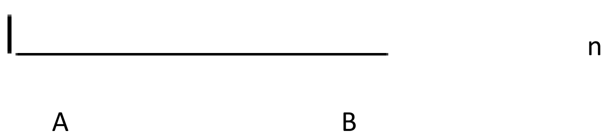
(i) Displacement

(ii) Velocity

(iii) Acceleration

REPRESENTATION OF A VECTOR

The vector quantity is always *described* by using *two capital letters* with respect to *arrow* on top or *small letters* with respect to *bar* at the bottom.



$\therefore\ \overrightarrow{AB}$ or $\underline{n}$

COMPONENTS OF VECTORS.

This depends on the dimension of vector as follows;

(i) For two dimensions namely as $\underline{i}$ and $\underline{j}$

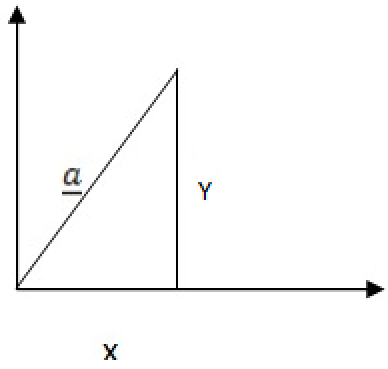
Where $\underline{i}$ =x- value

$\underline{j}$ = y – value

e.g. $\underline{a}$ = (x,y) coordinate form

$\underline{a} = x\underline{i} + y\underline{j}$ component form

Diagram



For three dimensions

Re: $(x, y, z \quad plane)$

This involves three components namely as $\underline{i}$, $\underline{j}$ and $\underline{k}$

Where,

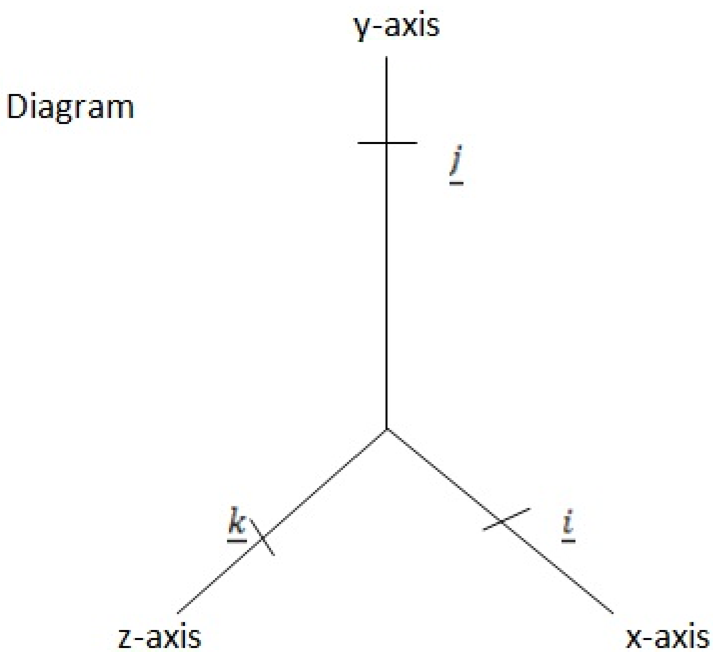
$\underline{i}$ = x – value

$\underline{j}$ = y – value

$\underline{k} = z \ - value$

$$\underline{r} = (x, y, z): coordinate\ form$$

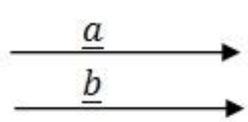
$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}: component\ form$$



TERMINOLOGIES APPLIED IN VECTOR ANALYSIS

1. PARALLEL VECTORS

These are vectors having the *same direction*.
e.g



2. EQUAL VECTORS

These are vectors having the *same magnitude* and *direction.*

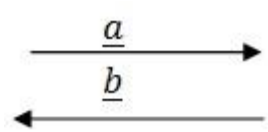
e.g.

$\underline{a}$ = 5N

$\underline{b}$ = 5N

3. NEGATIVE (OPPOSITE) VECTORS

These are vectors having the *same magnitude* but *opposite direction.*



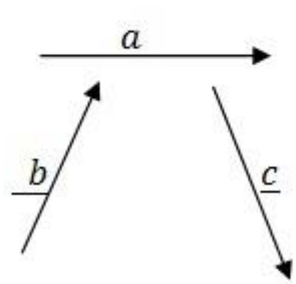
Hence

(i) $\underline{a} = -\underline{b}$ (vector in opposite direction

(ii) $\underline{a} = |\underline{b}|$ = b vector on the same direction

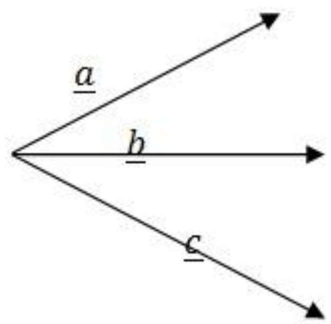
FREE VECTORS

These are vectors which *originate* from *different points.*



POSITION VECTORS

These are vectors which *originate* from the *same points.*



NB:

- Position vector of $\overrightarrow{AB} = \underline{b} - \underline{a}$
- Position vector of $\overrightarrow{AB} = \underline{c} - \underline{a}$
- Position vector of $\overrightarrow{OA} = \underline{a} - \underline{o}$
- Position vector of $\overrightarrow{OB} = \underline{b} - \underline{o}$

NULL VECTORS

These are vectors which have a *magnitude of zero* (length of zero) or

These are vectors which contain zero point

Eg.

$\underline{a}$ = (0,0)

$\underline{b}$ = (0,0,0)

COLLINEAR VECTORS

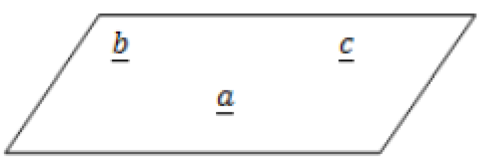
- These are vectors which *lie on the same line.*

i.e.

$\therefore \underline{a}$ *and* $\underline{b}$ *are collinear vectors*

COPLANAR VECTORS

These are vectors *which lie on the same plane.*



$\underline{a}$, $\underline{b}$ and $\underline{c}$ are coplanar vector

NB:

Consider the vector diagram below;

Where

A = initial (starting) point

B = Final (terminal) point

OPERATION IN VECTORS

These are

(i) Addition
(ii) Subtraction
(iii) Multiplication

i. ADDITION OF VECTORS

Suppose two dimensional vectors

$$\vec{A} = a_1 \underline{i} + b_1 \underline{j}$$

$$\vec{B} = a_2 \underline{i} + b_2 \underline{j}$$

$$\vec{A} + \vec{B} = a_1 i + b_1 j + a_2 i + b_2 j$$

$$\vec{A} + \vec{B} = (a_1 + a_2)\underline{i} + (b_1 + b_2)\underline{j}$$

Suppose three dimensional vectors

$$\vec{A} = a_1 \underline{i} + \underline{b_1} \underline{j} + C_1 \underline{k}$$

$$\vec{B} = a_2 \underline{i} + \underline{b_2} \underline{j} + C_2 \underline{k}$$

$$\vec{A} + \vec{B} = (a_1 + a_2)i + (\underline{b_1} + \underline{b_2})j + (C_1 + C_2)_k$$

RESULTANT VECTOR

Is the single vector which represents the effect of all vectors acting at a point.

Consider F_1, F_2, F_3 and F_4 are acting at a point

Hence

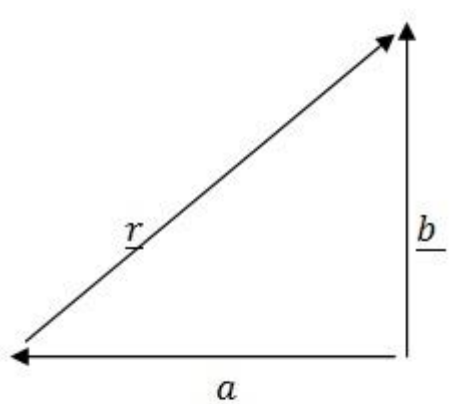
$$F = F_1 + F_2 + F_3 + F_4$$

Where F is the result force/ vector

LAWS OF VECTORS - ADDITION

A: TRIANGULAR LAW OF VECTORS APPLICATION

Consider the vector diagram below;



$\underline{a} + \underline{b} - \underline{r} = 0$

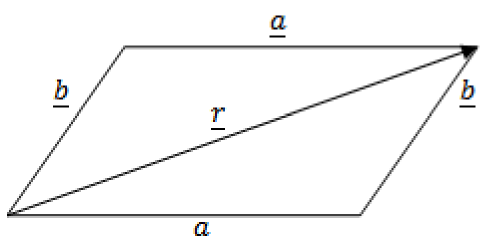
$\underline{r} = \underline{a} + \underline{b}$

$\underline{r} = \underline{a} + \underline{b}$

Where r is the resultant vector

B. PARALLELOGRAM LAW OF VECTORS ADDITION

- Consider the vector diagram below



$\underline{a} + \underline{b} - \underline{r} = 0$

$\underline{a} + \underline{b} = \underline{r}$……………………….(i)

$\underline{b} + \underline{a} - \underline{r} = 0$

$\underline{b} + \underline{a} = \underline{r}$ ………………………….(ii)

From (i) and (ii) above

$\therefore \underline{a} + \underline{b} = \underline{r} = \underline{b} + \underline{a}$

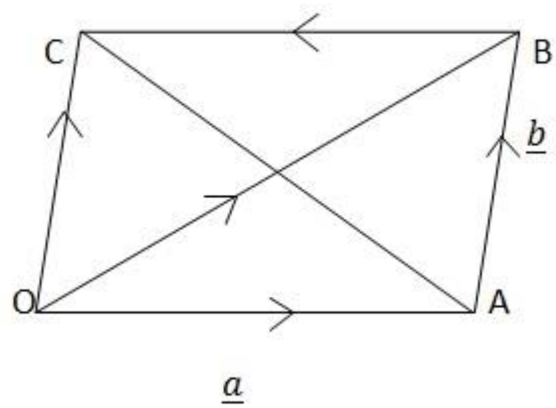
Proved

(ii) Addition of vectors is associative for any three vectors a, b and c

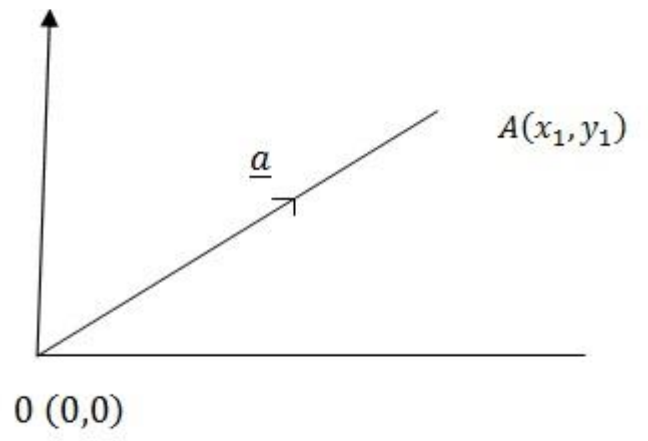
$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$

Proof

Consider the vector diagram below;



Individual but not considered



$\overrightarrow{OA} = a - 0$

(x_1, y_1) - $(0, 0)$

= $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

= $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$a + b + c = (\overrightarrow{AO} + \overrightarrow{AB}) + \overrightarrow{BC}$

= $\overrightarrow{OB} + \overrightarrow{BC}$

$\overrightarrow{OC}$ ......(i)

$a + (b + c) = \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{BC})$

$\overrightarrow{OA} + \overrightarrow{AC}$

$\overrightarrow{OC}$ ....(ii)

From (i) and (ii) above

= $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$

Proved

(iii) For….. of additive identity

For every vector a, we have;

$(\underline{a}+\underline{b})+\underline{c}=\underline{a}+(\underline{b}+\underline{c})$

Where;

$\longrightarrow 0$ L_1 The null (zero) vector

(iv) Entrance of addictive reverse

For every vector a we have

$a+(-a)=(-a)+a=0$

Where

$-a \rightarrow is\ the\ negative\ of\ vector$

$\underline{a}$ → is the positive of vector

$\underline{o}$ → is the null vector

ii. SUBTRACTION OF VECTORS

Suppose two dimensional vectors

$\vec{A}=a_1\underline{i}+b_1\underline{j}$

$\vec{B}=a_2\underline{i}+b_2\underline{j}$

Hence

$\bar{A} - \vec{B} = \left(a_1\underline{i} + b_1\underline{j}\right) - \left(a_2\underline{i} + b_2\underline{j}\right)$

$= a_1\underline{i} + b_1\underline{j} - a_2\underline{i} - b_2\underline{j}$

$= (a_1 i - a_2 i) + (b_1 j - b_2 j)$

$\vec{A} - \vec{B} = (a_1 - a_2)\underline{i} + (b_1 - b_2)\underline{j}$

- Suppose three dimensional vectors

$\vec{A} = a_1\underline{i} + b_1\underline{j} + C_1\underline{k}$

$\vec{B} = a_2\underline{i} + b_2\underline{j} + C_2\underline{k}$

Hence

$\bar{A} - \vec{B} = \left(a_1\underline{i} + b_1\underline{j} + C_1\underline{k}\right) - \left(a_2\underline{i} + b_2\underline{j} + C_2\underline{k}\right)$

$= a_1 i + b_1 j + C_1 k - a_2 i - b_2 j - C_2 k$

$\therefore (a_1 - a_2)\underline{i} + (b_1 - b_2)\underline{j} + (C_1 - C_2)\underline{k}$

Question 1

1. If $a = 2\underline{i} + \underline{j} + \underline{k}$ *and*

$b = 3\underline{i} - 2\underline{j} + 4\underline{k}$

(a Find (i) $a + b$

(ii) $b + a$

Comment of the results in (a) above

Question 2

Given that

$a = 3\underline{i} + 2\underline{j} + \underline{k}$

$b = 2\underline{i} - \underline{j} + 4\underline{k}$

(i) Find $a - b$

(ii) $b - a$

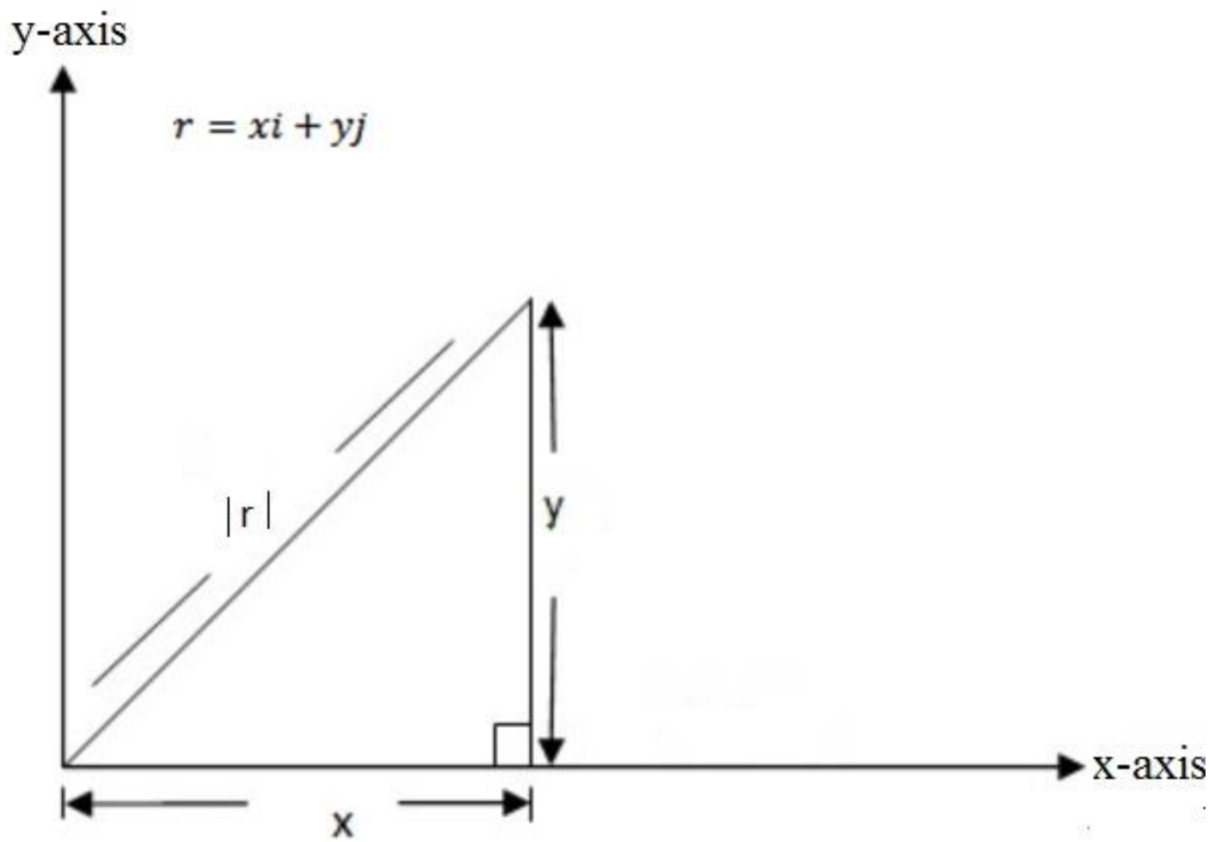
(b) Comment on the results in a above

MAGNITUDE OF A VECTOR

- The magnitude of a vector is a measure of length of the vector.

- - This is denoted by the symbol "| |"

(a) Consider two dimensional vector



By using Pythagoras theorem

Recall;

$x^2 + y^2 = |r|^2$

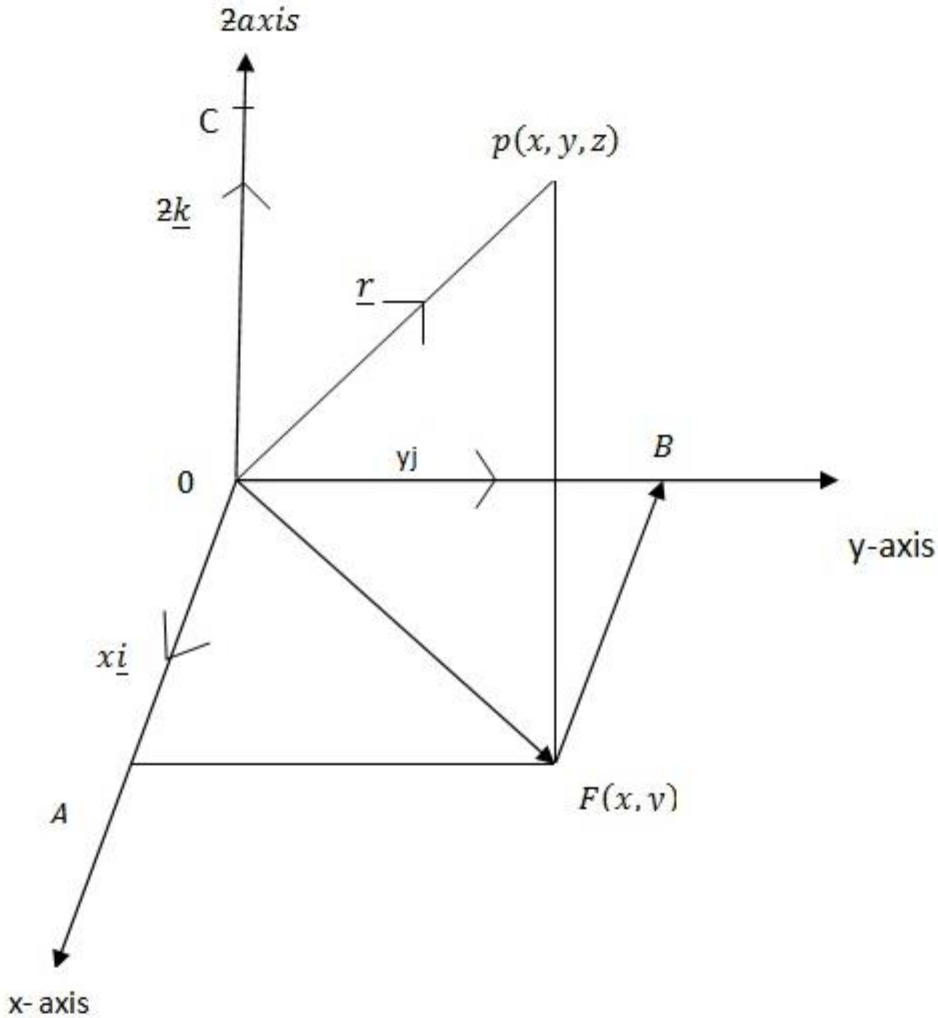
$|r|^2 = x^2 + y^2$

$\therefore \underline{r} = \sqrt{x^2 + y^2}$

Where

$|r|^2$ - is the magnitude/ module of the vector r

(b) Consider three dimensional vector $r = x\underline{i} + y\underline{j} + z\underline{k}$



RECTANGULAR RESOLUTION OF A VECTOR

Let: ox, oy, oz be three rectangular axes and i, j and k be three unit vectors parallel to x, y and z axes respectively.

If $\overrightarrow{OP} = r$ and the co – ordinates of $\boldsymbol{p}$ be $\overrightarrow{OA} = x\underline{i}, \overrightarrow{OB} = y\underline{j}, \overrightarrow{OC} = z\underline{k}$

Consider $\Delta\, OFP$

$\overrightarrow{OF} + \overrightarrow{FP} - \overrightarrow{OP} = 0$

$\overrightarrow{OF} + \overrightarrow{FP} = \overrightarrow{OP}$

$\overrightarrow{OP} = \overrightarrow{OF} + \overrightarrow{FP}$

$\overrightarrow{OP} = \left(\overrightarrow{OA} + \overrightarrow{AF}\right) + \overrightarrow{FP}$

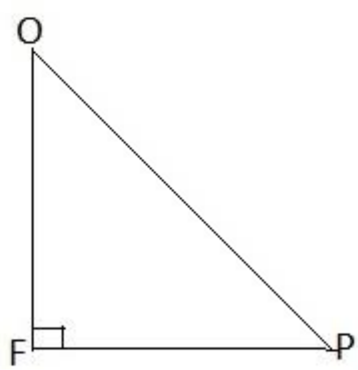
$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AF} + \overrightarrow{FP}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$

$\therefore\ \underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

Also consider the right angled OFP

i.e



Using Pythagoras theorem

i.e $a^2 + b^2 = c^2$

$\left(\overrightarrow{OF}\right)^2 + \left(\overrightarrow{FP}\right)^2 = \left(\overrightarrow{OP}\right)^2$

$\left(\overrightarrow{OP}\right)^2 = \left(\overrightarrow{OF}\right)^2 + \left(\overrightarrow{FP}\right)^2$

$\left(\overrightarrow{OP}\right)^2 = \left(\overrightarrow{OA}\right)^2 + \left(\overrightarrow{AF}\right)^2 + \left(\overrightarrow{FP}\right)^2$

$$(\overrightarrow{OP})^2 = (\overrightarrow{OA})^2 + (\overrightarrow{OB})^2 + (\overrightarrow{OC})^2$$

$$|r|^2 = x^2 + y^2 + z^2$$

$$\therefore\ |r| = \sqrt{x^2 + y^2 + z^2}$$

Where

$|r|$ = is the magnitude of the vector $r = x\underline{i} + y\underline{j} + z\underline{k}$

Question

Given that

$a = 2\underline{i} + \underline{j} + 2\underline{k}$

$b = 3\underline{i} + 3\underline{j} + \underline{k}$

Find $|a + b|$

DIRECTION RATIO AND DIRECTION COSINES

I. DIRECTION RATIO

Suppose the vector

$r = x\underline{i} + y\underline{j} + z\underline{k}$

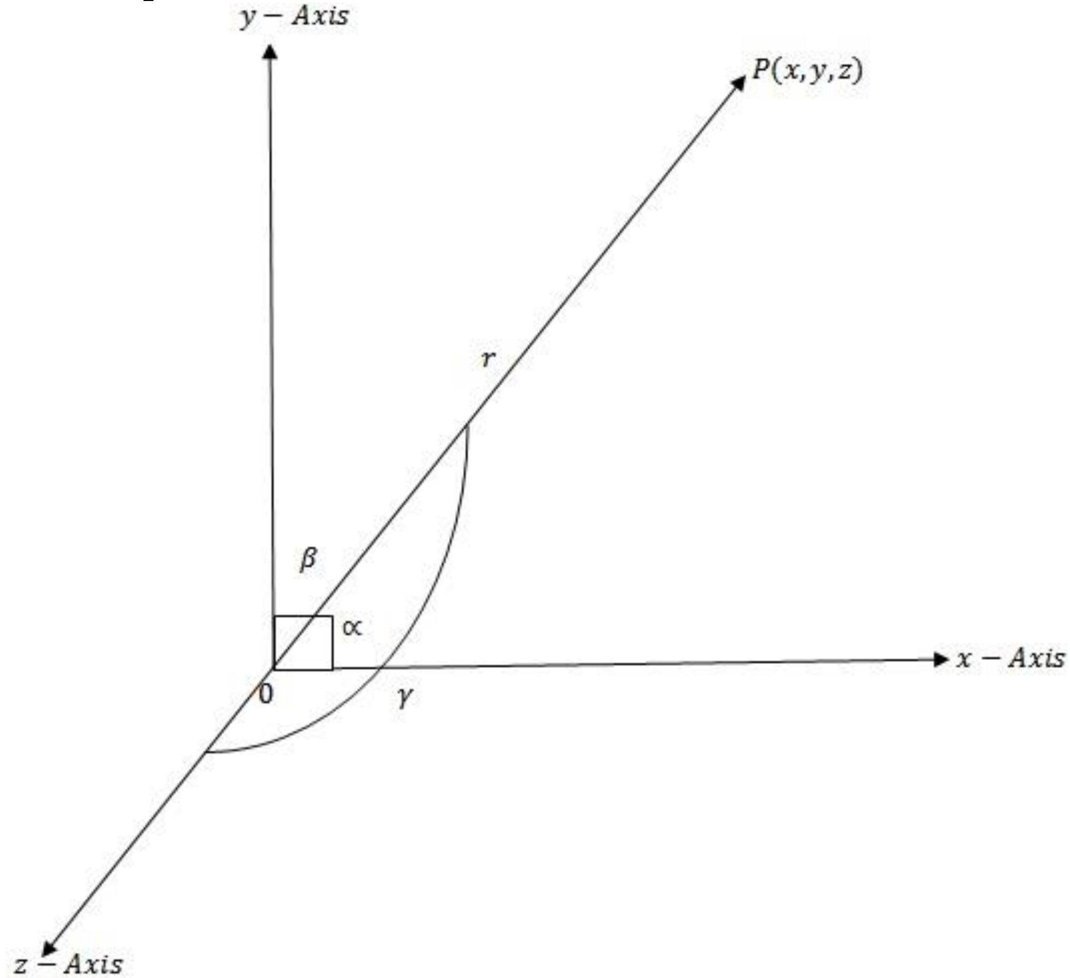
The direction ratio is given by

$Dr = x : y : z$

II. DIRECTION COSINE

Consider the vector

$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$



From three dimension plane.

$\overrightarrow{OP}$ Makes angles $\propto, B, and\ \gamma$ with $\underline{r}, \underline{j}\ and\ \underline{k}$ direction respectively

Hence

$\cos \propto = \frac{Adj}{Hyp}$

$\cos \propto = \frac{x}{\overrightarrow{OP}}$

$\cos \beta = \frac{Adj}{Hyp}$

$\cos\beta = \frac{y}{\overrightarrow{OP}}$

$\cos\gamma = \frac{z}{\overrightarrow{OP}}$

Therefore the direction cosines are

$\frac{x}{\overrightarrow{OP}}, \frac{y}{\overrightarrow{OP}}, \frac{z}{\overrightarrow{OP}}$

FACT IN DIRECTION COSINES

- Suppose the vector

$\underline{r} = x\underline{i} + y\underline{j} + 2\underline{k}$

$|r| = \sqrt{x^2 + y^2 + 2^2}$

Also the direction cosines are

$\cos\propto = \frac{x}{|r|}$

$\cos\beta = \frac{y}{|r|}$

$\cos\gamma = \frac{2}{|r|}$

Hence

$\cos^2\propto + \cos^2\beta + \cos^2\gamma = 1$

- The sum of square of the direction cosines is one.

Proof

i.e $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$|r| = \sqrt{x^2 + y^2 + z^2}$

Also

$$\cos \propto = \frac{x}{|r|}$$

$$(\cos \propto)^2 = \left(\frac{x}{|r|}\right)^2$$

$$\cos^2 \propto = \frac{x^2}{|r|^2} \text{ ---------------(i)}$$

$$\cos \beta = \frac{y}{|r|}$$

$$(\cos \beta)^2 = \left(\frac{y}{|r|}\right)^2$$

$$\cos^2 \beta = \frac{y^2}{|r|^2} \text{ ---------------------(ii)}$$

$$\cos \gamma = \frac{z}{|r|}$$

$$(\cos \gamma)^2 = \left(\frac{z}{|r|}\right)^2$$

$$\cos^2 \gamma = \frac{z^2}{|r|^2} \text{ ---------------------------(iii)}$$

Adding the equation (i) , (ii) and (iii)

$\cos^2 \propto + \cos^2\beta + \cos^2\gamma = \frac{x^2}{|r|^2} + \frac{y^2}{|r|^2} + \frac{z^2}{|r|^2}$

$= \frac{x^2+y^2+z^2}{|r|^2}$

But

$|r| = \sqrt{x^2+y^2+z^2}$

$|r|^2 = x^2+y^2+z^2$

$\cos^2 \propto + \cos^2\beta + \cos^2\gamma = \frac{x^2+y^2+z^2}{x^2+y^2+z^2}$

$\therefore \cos^2 \propto + \cos^2\beta + \cos^2\gamma = 1$

UNIT VECTOR

-Is the vector whose magnitude (modules) is one line a unit

-The unit vector in the direction of vector a is donated by read as " a cap" thus

NOTE:

Any vector can be compressed as the product of it's magnitude and it's unit vector i.e

$\hat{v} = \frac{v}{|v|}$

$\therefore v = |v| \, x \, \hat{v}$

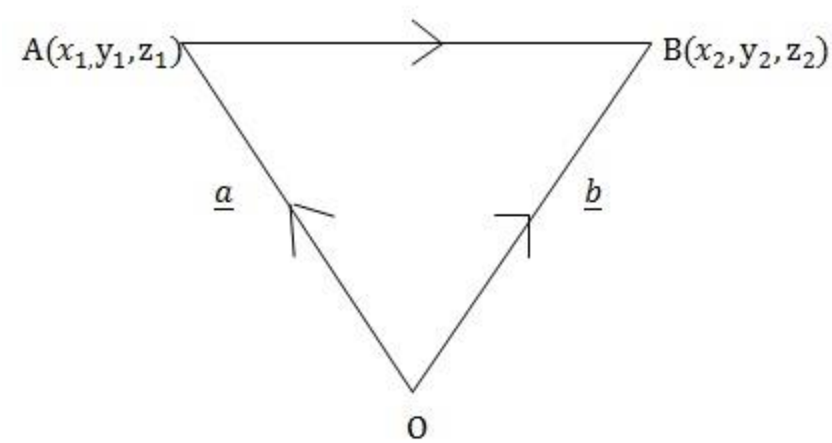
QUESTIONS

1. Find a vector in the direction of vector $5i - j + 2k$ which has a magnitude of 8 units

2. Find the direction ratio and direction cosines of the vector $\overrightarrow{OP}$ where p is the point (2, 3, -6)

THE FORMULA OF DISTANCE BETWEEN TWO POINTS

Suppose the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ whose position vectors are a and b respectively



$\underline{a} = \overrightarrow{OA} = x_1\underline{i} + y_1\underline{j} + z\underline{k}$

$\underline{b} = \overrightarrow{OB} = x_2\underline{i} + y_2\underline{j} + z_2\underline{k}$

HENCE

$\overrightarrow{OA} + \overrightarrow{AB} + -\overrightarrow{OB} = 0$

$\overrightarrow{OB} = \overrightarrow{OD} - \overrightarrow{OA}$

$\overrightarrow{AB} = b - a$

$\overrightarrow{AB} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} x_2 & - & x_1 \\ y_2 & - & y_1 \\ z_2 & - & z_1 \end{pmatrix}$

$\overrightarrow{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$

Hence

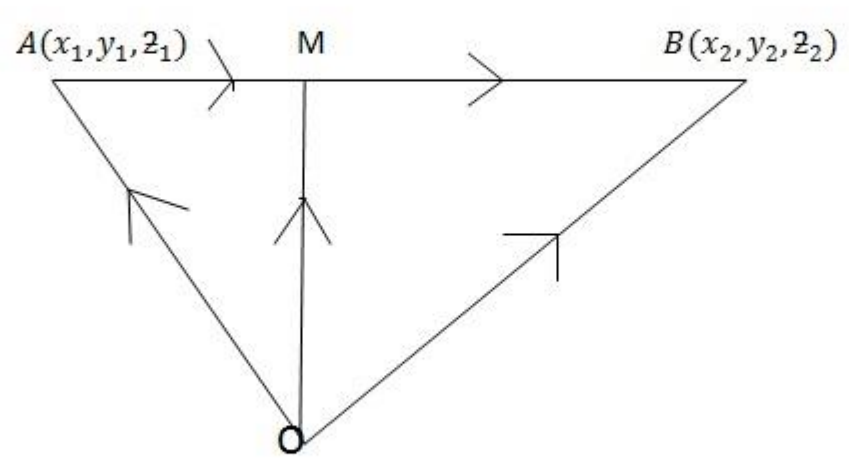
$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

-Formula distance between two point

MID POINT OF A LINE

Suppose M is the point which divide the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ whose position vectors are respectively a and b into two equal parts

i.e



a = $\overrightarrow{OA} = x_1 i + y_1 j + z_1 k$

b = $\overrightarrow{OB} = x_2 i + y_2 j + z_2 k$

Hence

$\overrightarrow{OA} + \overrightarrow{AM} + -\overrightarrow{OM}$

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} \quad [since\ m - is\ the\ mid - point]$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \tfrac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \tfrac{1}{2}\ \overrightarrow{OB} - \tfrac{1}{2}\ \overrightarrow{OA}$$

$$\overrightarrow{OM} = \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

$$\overrightarrow{OM} = \tfrac{1}{2}(\mathrm{a} + \mathrm{b})$$

$$\overrightarrow{OM} = \tfrac{1}{2}\left[\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right]$$

$$\overrightarrow{OM} = \tfrac{1}{2}\begin{bmatrix} x_1 & + & x_2 \\ y_1 & + & y_2 \\ z_1 & + & z_2 \end{bmatrix}$$

$$\overrightarrow{OM} = \left(\frac{x_1 + x_2}{2}\right)\underline{\mathrm{i}} + \left(\frac{y_1 + y_2}{2}\right)\underline{\mathrm{j}} + \left(\frac{z_1 + z_2}{2}\right)\underline{\mathrm{k}}$$

Therefore

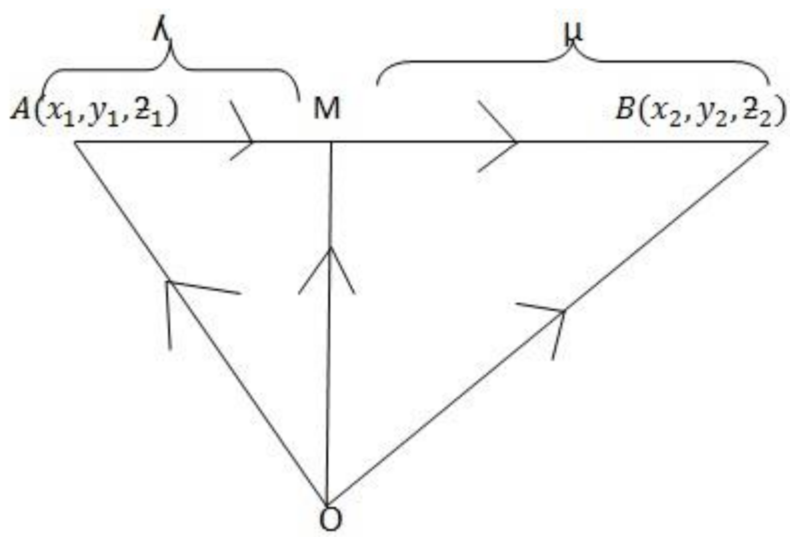
The co-ordinate of M is

$$m = \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}$$

INTERNAL AND EXTERNAL DIVISION OF A LINE (RATIO THEOREM)

I. INTERNAL DIVISION OF A LINE

- Suppose M- is the point which divides the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ whose point vectors are a and b respectively internally in the ratio X:ee



a = $\overrightarrow{OA}$ = $x_1\underline{i} + y_1\underline{j} + z_1\underline{k}$

b = $\overrightarrow{OB}$ = $x_2\underline{i} + y_2\underline{j} + z_2\text{k}$

$$\overrightarrow{OA} + \overrightarrow{AM} + -\overrightarrow{OM} = 0$$

$$\overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OM}$$

$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA}$ ……………(i)

$\overrightarrow{OM} + \overrightarrow{MB} + -\overrightarrow{OB} = 0$

$\overrightarrow{OM} + \overrightarrow{MB} = \overrightarrow{OB}$

$\overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM}$ ……………(ii)

By using ratio theorem

$$\frac{\overrightarrow{AM}}{\overrightarrow{MB}} = \frac{\lambda}{\mu}$$

$$\frac{\overrightarrow{OM}}{\overrightarrow{OB}} - \frac{\overrightarrow{OA}}{\overrightarrow{OM}} = \frac{\lambda}{\mu}$$

By using multiplication

$$\lambda(\overrightarrow{OB} - \overrightarrow{OM}) = \mu(\overrightarrow{OM} - \overrightarrow{OA})$$

$$\lambda\overrightarrow{OB} - \lambda\overrightarrow{OM} = \mu\overrightarrow{OM} - \mu\overrightarrow{OA}$$

$$\lambda\overrightarrow{OM} + \mu\overrightarrow{OM} = \lambda\overrightarrow{OB} + \mu\overrightarrow{OA}$$

$$\overrightarrow{OM}(\lambda + \mu) = \lambda\overrightarrow{OB} + \mu\overrightarrow{OA}$$

$$\overrightarrow{OM} = \frac{\lambda\overrightarrow{OB} + \mu\overrightarrow{OA}}{\lambda + \mu}$$

$$\overrightarrow{OM} = \frac{\lambda\underline{b} + \mu\underline{a}}{\lambda + \mu}$$

$$\overrightarrow{OM} = \frac{\lambda\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \mu\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}}{\lambda + \mu}$$

$$\overrightarrow{OM} = \frac{1}{\lambda + \mu}\begin{pmatrix} \lambda x_2 & + & \mu x_1 \\ \lambda y_2 & + & \mu y_1 \\ \lambda z_2 & + & \mu z_1 \end{pmatrix}$$

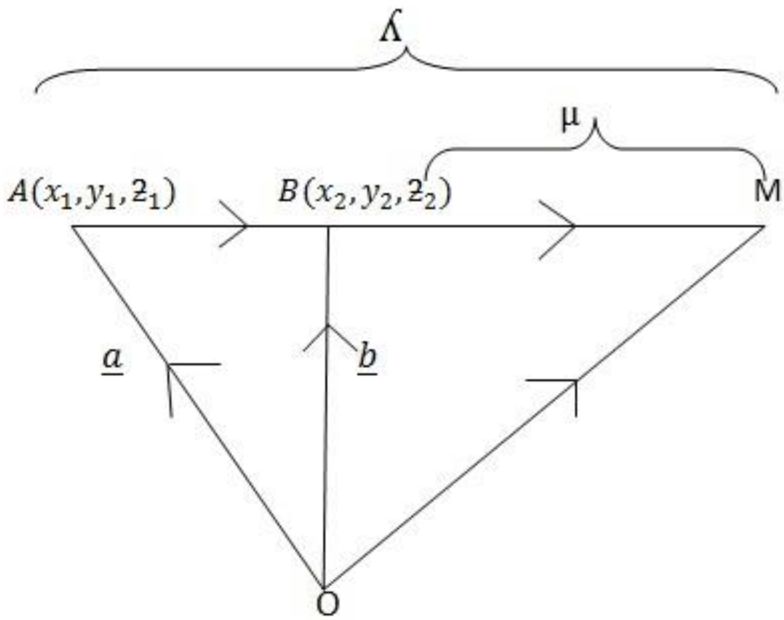
$$\overrightarrow{OM} = \left(\frac{\lambda x_2 + \mu x_1}{\lambda + \mu}\right)\underline{i} + \left(\frac{\lambda y_2 + \mu y_1}{\lambda + \mu}\right)\underline{j} + \left(\frac{\lambda z_2 + \mu z_1}{\lambda + \mu}\right)\underline{k}$$

The ordinate form of M is

$$m = \frac{\lambda x_2 + \mu x_1}{\lambda + \mu}, \frac{\lambda y_2 + \mu x_1}{\lambda + \mu}, \frac{\lambda z_2 + \mu z_1}{\lambda + \mu}$$

II. EXTERNAL DIVISION OF A LINE

Suppose M- is the point which divides the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ where position vectors are $\underline{a}$ and $\underline{b}$ respectively , externally is the ration $\lambda : ee$



$\underline{a} = \overrightarrow{AO} = x_1\underline{i} + y_1\underline{j} + z_1\underline{k}$

$\underline{b} = \overrightarrow{OB} = x_2\underline{i} + y_2\underline{j} + z_2k$

$\overrightarrow{OA} + \overrightarrow{AM} + -\overrightarrow{OM} = 0$

$\overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OM}$

$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA}$ ……………(i)

$\overrightarrow{OB} + BM + -\overrightarrow{OM} = 0$

$\overrightarrow{OB} + \overrightarrow{BM} = \overrightarrow{OM}$

$\overrightarrow{BM} = \overrightarrow{OM} - \overrightarrow{OB}$ ……………(ii)

By using ratio theorem

i.e

$$\frac{\overrightarrow{AM}}{\overrightarrow{BM}} = \frac{ʎ}{\mu}$$

$$\frac{\overrightarrow{OM}}{\overrightarrow{OM}} - \frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{ʎ}{\mu}$$

BY CROSSING MULTIPLICATION

$$ʎ(\overrightarrow{OM} - \overrightarrow{OB}) = \mu(\overrightarrow{OM} - \overrightarrow{OA})$$

$$\overrightarrow{ʎOM} - ʎ\overrightarrow{OB} = \mu\overrightarrow{OM} - \mu\overrightarrow{OA}$$

$$ʎ\overrightarrow{OM} - \mu\overrightarrow{OM} = ʎ\overrightarrow{OB} + \mu\overrightarrow{OA}$$

$$\overrightarrow{OM}(ʎ - \mu) = ʎ\overrightarrow{OB} - \mu\overrightarrow{OA}$$

$$\overrightarrow{OM} = \frac{ʎ\overrightarrow{OB} - \mu\overrightarrow{OA}}{ʎ - \mu}$$

$$\overrightarrow{OM} = \frac{ʎ\underline{b} - \mu\underline{a}}{ʎ - \mu}$$

$$\overrightarrow{OM} = \frac{ʎ\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \mu\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}}{ʎ - \mu}$$

$$\overrightarrow{OM} = \frac{1}{ʎ - \mu}\begin{pmatrix} ʎx_2 & - & \mu x_1 \\ ʎy_2 & - & \mu y_1 \\ ʎz_2 & - & \mu z_1 \end{pmatrix}$$

$$\overrightarrow{OM} = \left(\frac{\lambda x_2 - \mu x_1}{\lambda - \mu}\right)\underline{i} + \left(\frac{\lambda y_2 - \mu x_1}{\lambda - \mu}\right)\underline{j} + \left(\frac{\lambda z_2 - \mu z_1}{\lambda - \mu}\right)\underline{k}$$

Therefore

The co-ordinate of M

$$m = \left(\frac{\lambda x_2 - \mu x_1}{\lambda - \mu}\right), \left(\frac{\lambda y_2 - \mu x_1}{\lambda - \mu}\right), \left(\frac{\lambda z_2 - \mu z_1}{\lambda - \mu}\right)$$

External division of a line where $\lambda \neq \mu$

QUESTIONS

1.Find the length of the line $\overrightarrow{AB}$ of $A(2,7,-1)$ and $B(4,1,2)$

2. Find the position vector which divides line $\overrightarrow{AB}$ having point $A(2,1,3)$ and $B(3,1,1)$ into two equal points.

3. A and B are two points whose vectors are $3\underline{i} + \underline{j} - 2\underline{k}$ and $\underline{i} - 3\underline{j} - \underline{k}$ respectively. Find the position vector of the points dividing AB.

(a) Internally in the ratio 1:3

(b) Externally in the ratio 3:1

III. MULTIPLICATION OF A VECTOR

(A) SCALAR MULTIPLICATION OF A VECTOR

In this case a vector is multiplied by a certain constant called scalar

Let

$\vec{A} = a_1\underline{i} + b_1\underline{j} + C_1\underline{k}$

$t\vec{A} = ta_1\underline{i} + tb_1\underline{j} + tC_1\underline{k}$

Where

t = is a scalar

(B) Suppose we have two vectors say $\underline{a}$ *and* $\underline{b}$

i.e

$\underline{a}$ *dot* $\underline{b} \equiv \underline{a} \cdot \underline{b}$

$\overrightarrow{A} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$

$\vec{B} = a_2\underline{i} + b_2\underline{j} + C_2\underline{k}$

THEREFORE

$$\vec{A} \cdot \vec{B} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

$$\therefore \vec{A} \cdot \vec{B} = a_1a_2 + b_1b_2 + C_1C_2$$

QUESTIONS

If $\underline{a} = 2\underline{i} + 3\underline{j} + 4\underline{k}$ and

$\underline{b} = 7\underline{i} + 2\underline{j} - 3\underline{k}$

(a) Find (i) $\underline{a} \cdot \underline{b}$

(ii) $\underline{b} \cdot \underline{a}$

(b) Comment on results in $\underline{a}$ above

DEFINITION OF DOT PRODUCT

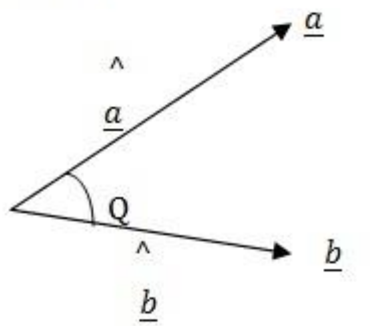
For vectors $\underline{a}$ and $\underline{b}$

$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|Q$$

Where

$Q -$ is the above between $\underline{a}$ and $\underline{b}$

Diagram



$$\hat{\underline{a}}.\hat{\underline{b}} = \cos Q$$

$$\frac{\underline{a}}{|\underline{a}|}.\frac{\underline{b}}{|\underline{b}|} = \cos Q$$

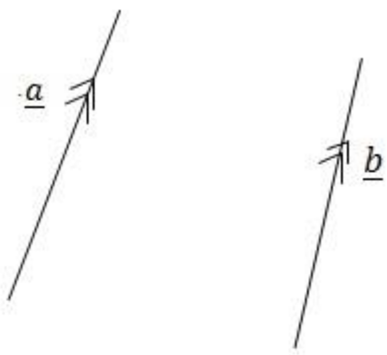
$$\underline{a}.\underline{b} = |\underline{a}|\ |\underline{b}| \cos Q$$

$$\boxed{\underline{a}.\underline{b} = |\underline{a}|\ |\underline{b}| \cos Q}$$

<u>CHARACTERISTICS</u>

1. 1. PARALLEL VECTOR

Two vector are said to be parallel of the angle between them is zero



Mathematically

From

$\underline{a}.\underline{b} = \underline{|a|}\ \underline{|b|}$ = cos Q

But Q = 0^0

$\underline{a}.\underline{b} = \underline{|a|}\ \underline{|b|}$ = cos Q

$\underline{a}.\underline{b} = \underline{|a|}\ \underline{|b|}$

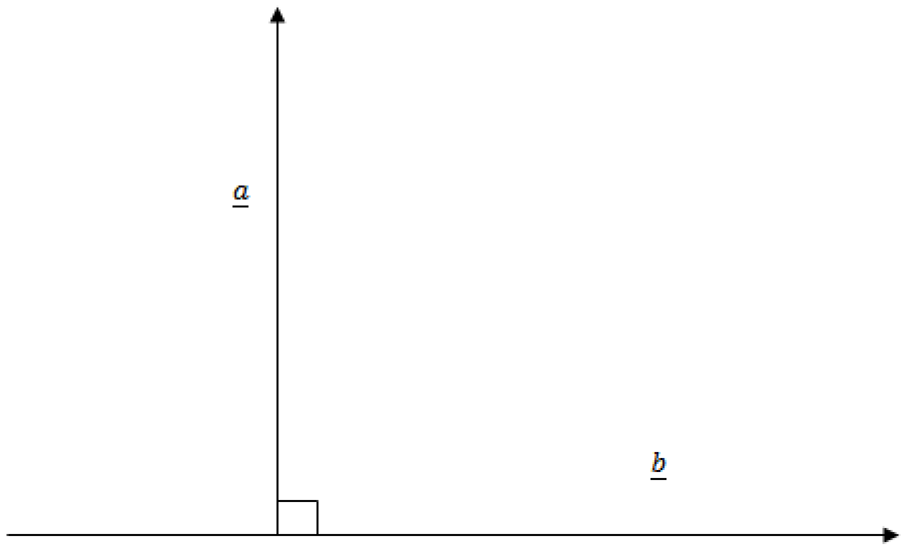
$\underline{a}.\underline{b} = \underline{|Q|}\ \underline{|b|}$

This is one among the

2. 2. Orthogonal vectors

Two vectors are said to be orthogonal of the angle between them is 90°

Mathematically



From

$$\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| = cosQ$$

But Q = 90° (orthogonal or perpendicular vector)

$\underline{a}.\underline{b} = \underline{|a|}\ \underline{|b|}$ = cos 90

$\underline{a}.\underline{b} = 0$

This is conditional for the orthogonal vector

THEOREM.

(a)For the definition

$\underline{a}.\underline{b} = \underline{|a|}\ \underline{|b|}\cos\theta$

$\underline{i}.\underline{i} = \underline{|i|}\ \underline{|i|}$

$\underline{i}$=(1,0)

$\underline{j}$=(1,0)

$\underline{i}.\underline{i} = 1$

Therefore

$\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$

From the definition

$\underline{a}.\underline{b} = \underline{|a|}\ \underline{|b|}\cos\theta$

$\underline{i}.\underline{j} = \underline{|i|}\ \underline{|j|}\cos 90$

$\underline{i}.\underline{j} = |x| \times 0$

$\underline{i}.\underline{j} = 0$

Therefore

$\underline{i}.\underline{j} = \underline{i}.\underline{k} = \underline{j}.\underline{k} = 0$

- Suppose the vector

$\vec{A} = a_1\underline{i} + b_1\underline{j} + C_1\underline{k}$

$\vec{B} = a_2\underline{i} + b_2\underline{j} + C_2\underline{k}$

$\vec{A}.\vec{B} = \left(a_1\underline{i} + b_1\underline{j} + C_1\underline{k}\right).\left(a_2\underline{i} + b_2\underline{j} + C_2\underline{k}\right)$

$= a_1a_2\underline{i}.\underline{i} + a_1b_2\underline{i}.\underline{j} + a_1C_2\underline{i}.\underline{k} + b_1a_2\underline{j}.\underline{i} + b_1b_2\underline{j}.\underline{j} + b_1C_2\underline{j}.\underline{k} + C_1a_2\underline{k}.\underline{i} + C_1b_2\underline{k}.\underline{j} + C_2C_1\underline{k}.\underline{k}$

$\therefore\ \vec{A}.\vec{B} = a_1a_2 + b_1b_2 + C_1C_2$

From the definition

$\underline{a}.\underline{b} = |\underline{a}|\,|\underline{b}| = \cos\theta$

$\underline{a}.\underline{a} = |\underline{a}|\,|\underline{a}| = \cos\theta$

$\underline{a}.\underline{a} = |\underline{a}|\,|\underline{a}|$

$\underline{a}.\underline{a} = |\underline{a}|^2$

- Consider the vector

$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$\underline{r}.\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\underline{r}^2 = x^2 + y^2 + z^2$

$\underline{r}^2 = |\underline{r}|^2$

$|\underline{r}|^2 = x^2 + y^2 + z^2$

$|\underline{r}|^2 = \sqrt{x^2 + y^2 + z^2}$

QUESTIONS

1. If $\underline{a} = \underline{i} + 2\underline{j} + 2\underline{k}$ and $\underline{b} = 2\underline{i} + 3\underline{j} + 4\underline{k}$

Find the angle between $\underline{a}$ and $\underline{b}$

2. Show that the vectors
$\underline{a} = \underline{i} - 2\underline{j} - 2\underline{k}$ and
$\underline{b} = 2\,\underline{i} + 4\underline{j} + 5\underline{k}$ are orthogonal

3. If $\overline{|a|}$=2, $\overline{|b|}$=3
$\theta = 60°$, find $|\underline{a} + \underline{b}|$

4. The vectors $p(-5,7,1)$ and
$q(k,k,k)$ where k> 0 are such that
$\underline{p} + \underline{q}$ and $\underline{p} - \underline{q}$ are orthogonal find k

5. If $\underline{a} = 2\,\underline{i} + \lambda\,\underline{j} + 3\underline{k}$ and
$\underline{b} = 4\,\underline{i} + 2\underline{j} + \lambda\,\underline{k}$

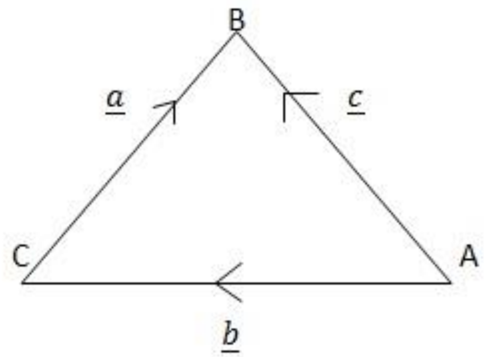
Find the value of λ if $\underline{a}$ and $\underline{b}$ are orthogonal

6.If $\overline{|a|}$=2, $\overline{|b|}$=3, $\theta = 60°$
Find $|\underline{a} + 2\underline{b}|$

APPLICATION OF DOT PRODUCT

1. TO VERIFY COSINE RULE

Consider the vector diagram below



$\underline{b} + \underline{a} + \underline{-c} = 0$

$\underline{a} + \underline{b} = \underline{c}$

$\underline{b} = \underline{c} - \underline{a}$......................(i)

Dot by $\underline{b}$ *on both side of equation .... (i)above*

$\underline{b} = \underline{c} - \underline{a}$

$\underline{b} . \underline{b} = (\underline{c} - \underline{a}).\underline{b}$

$\underline{b}^2 = (\underline{c} - \underline{a})(\underline{c} - \underline{a})$

$\underline{b}^2 = c^2 - 2\underline{a}\,\underline{c} + a^2$

$\underline{b}^2 = a^2 + \underline{c}^2 - 2\underline{a}\,\underline{c}$

$|\underline{b}|^2 = |\underline{a}|^2 + |\underline{c}|^2 - 2\,|\underline{a}|\,|\underline{c}| \cos \hat{B}$

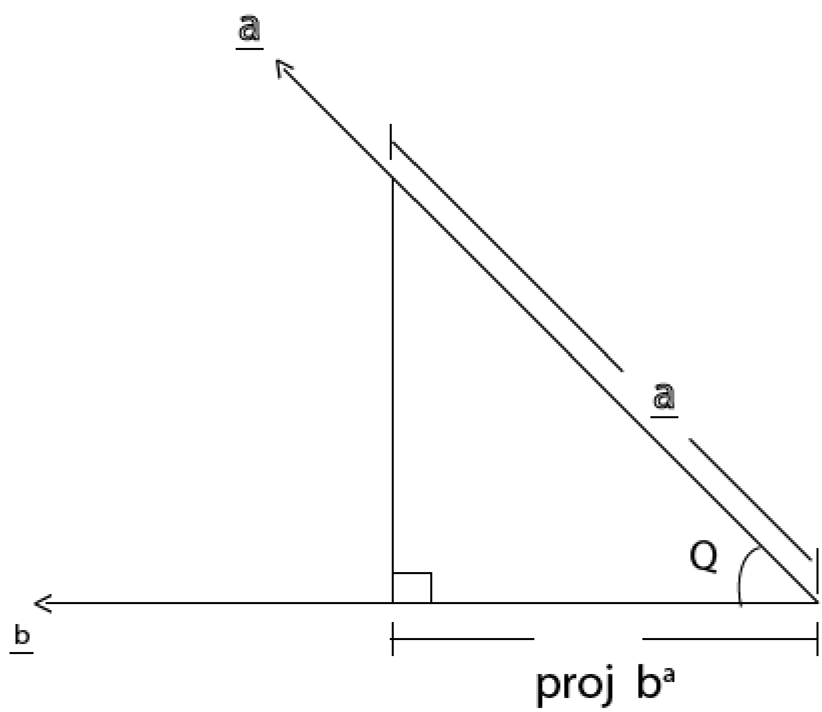
Therefore

$|\underline{b}|^2 = |\underline{a}|^2 + |\underline{c}|^2 - 2\,|\underline{a}|\,|\underline{c}|\cos\hat{B}$

02. USED TO FIND THE PROJECTION OF ONE VECTOR ONTO ANOTHER VECTOR

- Suppose the projection of a onto b

i.e



Cos Q = $\frac{Adj}{Hyp}$

Cos Q = $\frac{proj\,b^{a}}{|\underline{a}|}$

proj b^{a} = $|\underline{a}|$ cos Q ---------– i

Also

$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|$ cos Q

$\frac{\underline{a}\cdot\underline{b}}{|\underline{b}|} = |\underline{a}|$ cos Q ---------–ii

Equalizing i and ii as follows;

proj $\underline{b}^{\underline{a}} = |\underline{a}|$ cos Q = $\frac{\underline{a}\cdot\underline{b}}{|\underline{b}|}$

proj $\underline{b}^{\underline{a}} = \frac{\underline{a}\cdot\underline{b}}{|\underline{b}|}$

similary

proj $\underline{b}^{\underline{a}} = \frac{\underline{a}\cdot\underline{b}}{|\underline{b}|}$

Where;

Proj $\underline{b}^{\underline{a}}$ = projection of $\underline{a}$ onto $\underline{b}$

Proj $\underline{a}^{\underline{b}}$ = projection of $\underline{b}$ onto $\underline{a}$

VECTOR PROJECTION

This is given by

V proj$\underline{b}^{\underline{a}}$ = $\left[\frac{\underline{a}.\underline{b}}{|\underline{b}|}\right]$. $\frac{\underline{b}}{|\underline{b}|}$

And

V. proj$\underline{a}^{\underline{b}}$ = $\left[\frac{\underline{a}.\underline{b}}{|\underline{a}|}\right]$. $\frac{\underline{a}}{|\underline{a}|}$

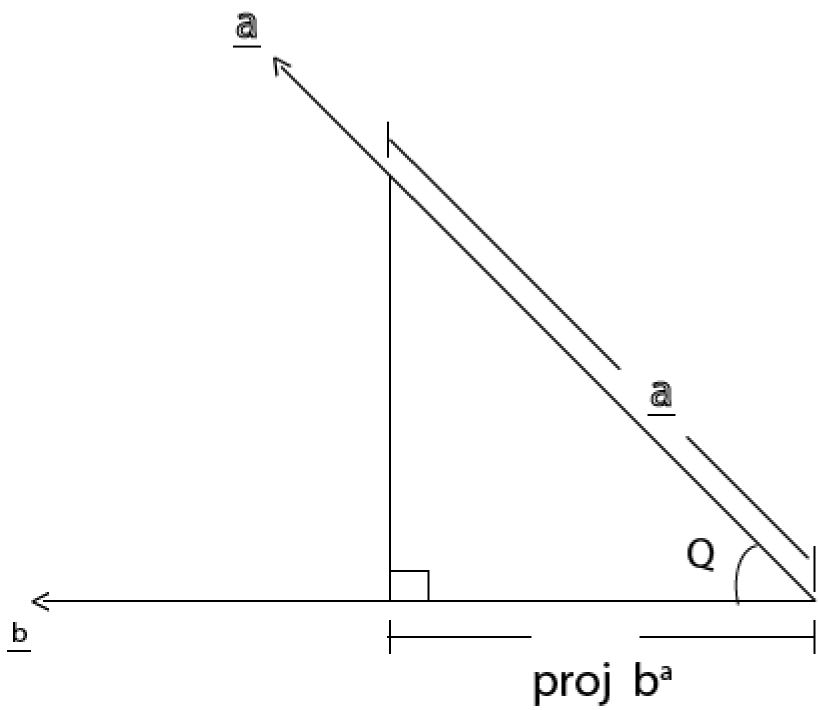
Where;

V. proj $\underline{b}^{\underline{a}}$ = vector projection of $\underline{a}$ onto $\underline{b}$

V. proj $\underline{a}^{\underline{b}}$ = vector projection of $\underline{b}$ onto $\underline{a}$

03. TO FIND THE WORKDONE

- Consider the diagram below



Force applied (F) = component of tone

cos Q = $\frac{Adj}{Hypo}$

Cos Q = $\frac{F}{|\underline{F}|}$

F = $|\underline{F}|$ Cos Q

Also

Distance d = $|displacement, \underline{d}|$

d = $|\underline{d}|$ ……….ii

Hence

Work done = Force applied (F) x distance (d)

W.D = F x d

W. D = $|\underline{F}|$ cos Q x $|\underline{d}|$

= $|\underline{F}||\underline{d}|$ cos Q

= $\underline{F} . \underline{d}$

W.D = $|\underline{F} . \underline{d}|$

Note

i) Force F in the direction of vector $\underline{a}$

Force applied = F.$\frac{\underline{a}}{|\underline{a}|}$

ii) Distance in the direction of vector $\underline{b}$

Displacement $\underline{d}$ = d.$\frac{\underline{b}}{\underline{b}}$

Individual

i) $\underline{F}$ = 2i + $\underline{j}$ + 2$\underline{k}$

$\underline{d}$ = i + $\underline{j}$ + $\underline{k}$

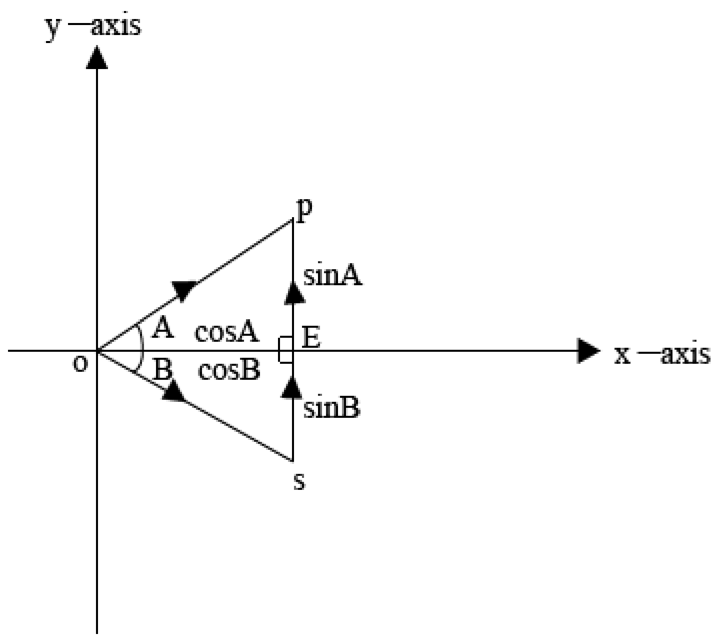
W.D = $|\underline{F} . \underline{d}|$

04. TO PROVE COMPOUND ANGLE FORMULA OF COSINE

i.e cos (A + B) = cos A cos B – sin A sin B

- consider the vector diagram below.

Diagram



$\overrightarrow{op}$ = (cos A) $\underline{i}$ + (sin A) $\underline{j}$

$\overrightarrow{os}$ = (cos B) $\underline{i}$ - (sin B) $\underline{j}$

Hence

$\overrightarrow{op} \cdot \overrightarrow{os} = |\overrightarrow{op}||\overrightarrow{os}|$ cos (A + B)

but

$\overrightarrow{op} \cdot \overrightarrow{os} = \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \begin{pmatrix} \cos B \\ -\sin B \end{pmatrix}$

= cos A cos B + -sin A sin 0

= cos A cos B – sin A sin B

Also

$|\overrightarrow{op}| = \sqrt{(\cos A)^2 + (\sin A)^2}$

$= \sqrt{cos\ ^2A + sin^2 A}$

$= \sqrt{1}$

= 1

$|\overrightarrow{os}| = \sqrt{(\cos B)^2 + (-\sin B)^2}$

$= \sqrt{cos\ ^2B + sin^2 B}$

$= \sqrt{1}$

= 1

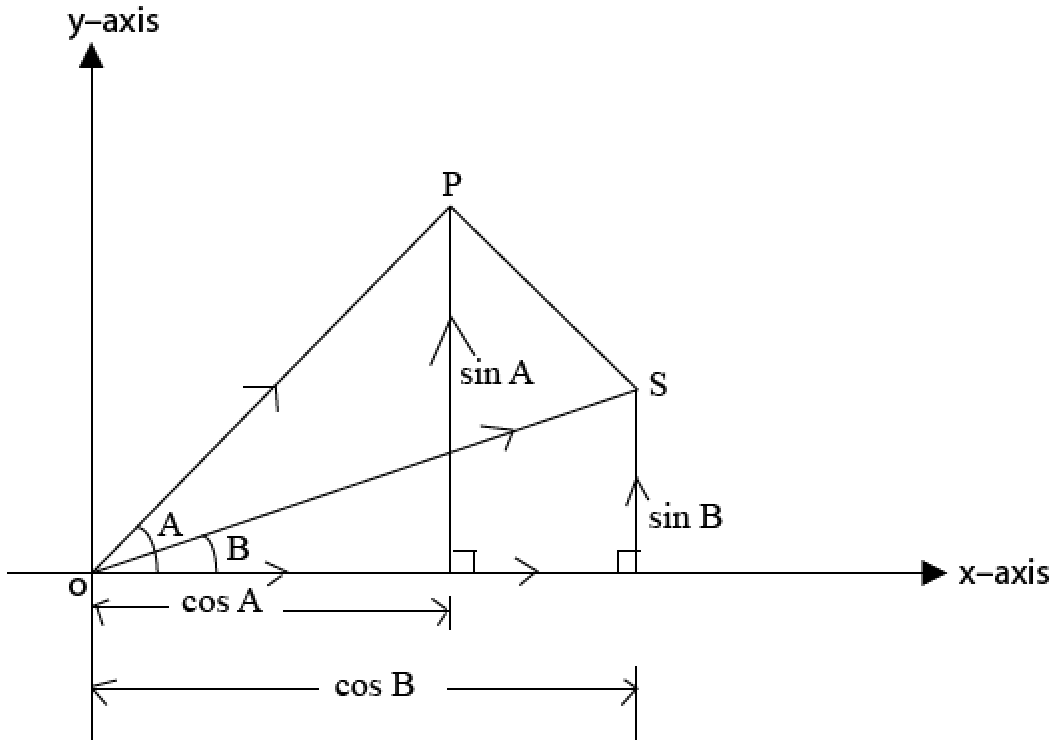
Therefore

Cos A cos B – sin A sin B = (1)(1) cos (A + B)

Cos (A + B) = cos A cos B – sin A sin B

Proved

Pg. 2 drawing



$\overrightarrow{op}$ = (cos A) $\underline{i}$ + (sin A) j

$\overrightarrow{os}$ = (cos B) I + (sin B) $\underline{j}$

Hence

$\overrightarrow{op} . \overrightarrow{os} = |\overrightarrow{op}|\ |\overrightarrow{os}|$ cos (A – B)

But

$\overrightarrow{op} . \overrightarrow{os} = \binom{\cos A}{\sin A} \binom{\cos B}{\sin B}$

$\overrightarrow{op} . \overrightarrow{os}$ = cos A cos B + sin A sin B

Also

$\overrightarrow{op} = \sqrt{(\cos A)^2 + (\sin A)^2}$

$= \sqrt{cos\ ^2A + sin^2A}$

$= \sqrt{1}$

= 1

$|\overrightarrow{os}| = \sqrt{(\cos B)^2 + (\sin B)^2}$

$= \sqrt{cos\ ^2B + sin^2B}$

$= \sqrt{1}$

= 1

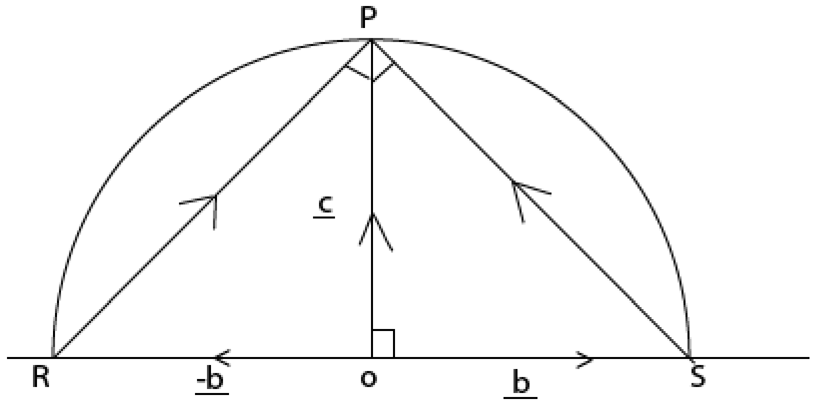
There
Cos A cos B + sin A sin B = (1) (1) cos (A – B)

Cos (A – B) = cos A cos B + sin A sin B
Proved

06. TO PROVE THAT AN INSCRIBED ANGLE SUBTENDING A SEMI – CIRCLE IS A RIGHT ANGLE

- Consider the vector diagram below.

Pg. 2 drawing



To prove that

$< S\hat{P}R = 90^0$

$\overrightarrow{RP} . \overrightarrow{SP} = 0$

$-b + \overrightarrow{RP} + - \underline{c} = 0$

$- \underline{b} + \overrightarrow{RP} = \underline{c}$

$\overrightarrow{PR} = \underline{b} + \underline{c}$

$\underline{b} + \overrightarrow{sp} + - \underline{c} = 0$

$\underline{b} + \overrightarrow{sp} = \underline{c}$

$\overline{sp} = \underline{c} - \underline{b}$

Hence

$\overrightarrow{RP} . \overrightarrow{SP} = (\underline{c} + \underline{b}) (\underline{c} - \underline{b})$

$\overrightarrow{RP} . \overrightarrow{SP} = (\underline{c})^2 - (\underline{b})^2$

$\overrightarrow{RP} \cdot \overrightarrow{SP} = \underline{c}_2 - \underline{b}_2$

$\overrightarrow{RP} \cdot \overrightarrow{SP} = |\underline{c}|_2 - |\underline{b}|_2$

But

$|\underline{c}| = |\underline{b}| =$ radius, r

$\overrightarrow{RP} \cdot \overrightarrow{SP} = 0$

$\Delta S\hat{P}R = 90°$

Proved

QUESTION

17. Find the projection of $\underline{i} + 2\underline{j} - 3\underline{k}$ onto $\underline{i} + 2\underline{j} + 2\underline{k}$

18. Find the vector projection of $\underline{a}$ onto $\underline{b}$. If $\underline{a} = 2\underline{i} + 2\underline{j} + \underline{k}$ and $\underline{b} = 3\underline{i} + \underline{j} + 2\underline{k}$

19. Find the work done of the force of (2i + 3i + k) n s pulling a load (3i + j k) m

20. Find the work done of the force of (2i + 3j + k) N is pulling a load a distance of 2m in the direction of 2m in the direction of

$\underline{a} = 3\underline{i} + 2\underline{j} + 2\underline{k}$

21. Find a vector which has magnitude of 14 in the direction of $2\underline{i} + 3\underline{j} + \underline{k}$

CROSS (VECTOR) PRODUCT (X or $\wedge$)

$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin \emptyset . \hat{n}$

Where

$\hat{n}$ - is the unit vector perpendicular to both vector $\underline{a}$ and $\underline{b}$

$\underline{n} = \underline{a} \times \underline{b}$

$\underline{\hat{n}} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

Hence

$\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \emptyset . \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$1 = \frac{|\underline{a}| |\underline{b}| \sin \emptyset}{|\underline{a} \times \underline{b}|}$

$|\underline{a} \times \underline{b}| = |\underline{a}| |\underline{b}| \sin \emptyset$

Therefore

$|\underline{a} \times \underline{b}| = |\underline{a}| |\underline{b}| \sin \emptyset$

OR

$|\underline{a} \wedge \underline{b}| = |\underline{a}| |\underline{b}| \sin \emptyset$

Where

$\emptyset$ – is the angle between the vector $\underline{a}$ and $\underline{b}$

Again

Suppose the vector

$\underline{V_1} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$

$\underline{v_2} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$

Hence

$v_1 + v_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \end{vmatrix}$

$v_1 + v_2 = \underline{i}\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - j\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \underline{k}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Note

i) If you cross two vectors, the product is also the vector.

ii) Cross (vector) product uses the knowledge of determinant of 3 x 3 matrix.

iii) From the definition.

$|\underline{a}\ x\ \underline{b}| = |\underline{a}|\ |\underline{b}| \sin Ø$

$|\underline{i}\ x\ \underline{i}| = |\underline{i}|\ |\underline{i}| \sin 0^0$

Individual

$\underline{i}$ = (1, 0, 0)

j = (0, 1, 0)

k = (0, 0, 1)

$|\underline{i} \ x \ \underline{i}|$ = (1) (1) (0)

$|\underline{i} \ x \ \underline{i}|$ = 0

$|\underline{i} \ x \ \underline{i}|$ = 0

Hence

i x i = j x $\underline{j}$ = $\underline{k}$ x $\underline{k}$ = 0

iv) From the definition

$$|\underline{a} \ x \ \underline{b}| = |\underline{a}| \ |\underline{b}| \sin Ø$$

$$|\underline{i} \ x \ \underline{j}| = |\underline{i}| \ |\underline{j}| \sin 90$$

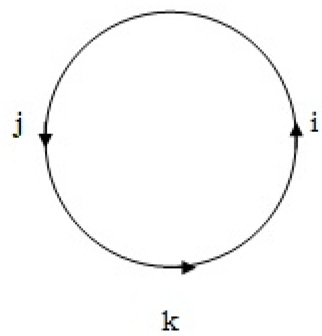
$$|\underline{a} \ x \ \underline{b}| = |\underline{a}| \ |\underline{b}| \sin Ø$$

$$|\underline{i} \ x \ \underline{j}| = 1$$

$\underline{i}$ x $\underline{j}$ = k

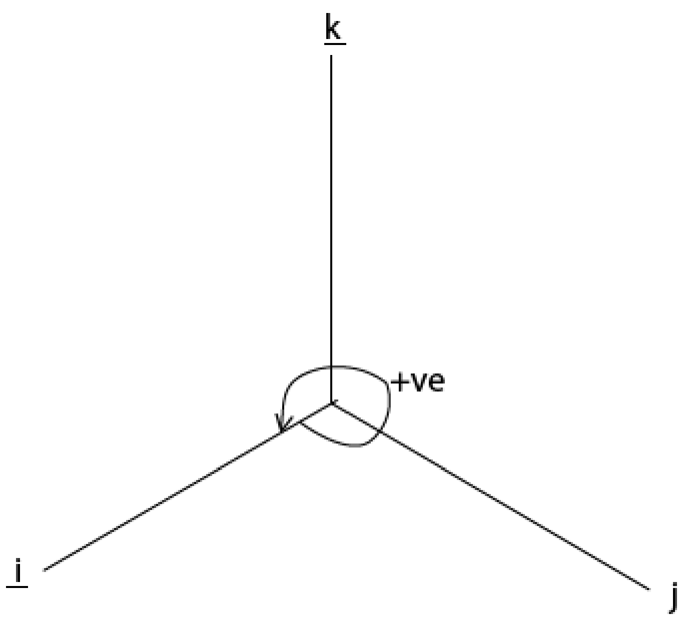
Generally

Consider the component vector



For anticlockwise (+ve)

Pg. 4 drawing

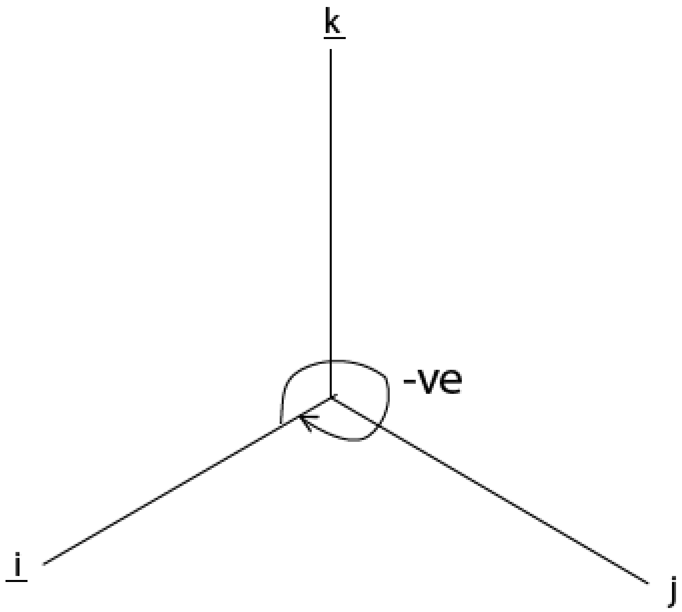


i) $\underline{i} \times \underline{j} = \underline{k}$

ii) $\underline{j} \times \underline{k} = \underline{i}$

iii) $\underline{k} \times \underline{j} = \underline{j}$

For clockwise (-ve)



i) $\underline{i} \times \underline{k} = \underline{k}$

ii) $\underline{k} \times \underline{j} = -\underline{i}$

iii) $\underline{k} \times \underline{i} = -\underline{k}$

THEOREM

From the definition

$$|\underline{a} \ x \ \underline{b}| = |\underline{a}| \ |\underline{b}| \sin Ø$$

$$|\underline{a} \ x \ \underline{b}| = |\underline{a}| \ |\underline{b}| \sin Ø$$

\- $$|\underline{a} \ x \ \underline{a}| = |\underline{a}| \ |\underline{a}| \sin Ø$$

$$|\underline{a} \; x \; \underline{a}| = |\underline{a}| \, |a| \sin Ø$$

$$|\underline{a} \; x \; \underline{a}| = 0$$

$\underline{a} \; x \; \underline{a} = 0$

Is the condition for collinear (parallel) vectors

a) $\underline{a} \; x \; \underline{a} \neq \underline{b} \; x \; \underline{a}$

$= - (\underline{b} \; x \; \underline{a})$

Questions

22. If $\underline{a} = 2\underline{i} + 6\underline{j} + 3\underline{k}$ and $\underline{b} = \underline{i} + 2\underline{j} + 2\underline{k}$. Find the angle between $\underline{a}$ and $\underline{b}$

23. Determine a unit vector perpendicular to $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$ and $\underline{b} = 4\underline{j} + 3\underline{j} - \underline{k}$

24. If $\underline{a} = 2\underline{i}$ + j + 2k and $\underline{b} = 3\underline{i} + 2\underline{j} + \underline{k}$

Find $\underline{b}$ â‹€$\underline{Q}$

Box product

-This involves both cross and dot product

Suppose $\underline{a} . \underline{b} \; x \; \underline{a}$ then start with cross (x) followed by DOT (.)

$\underline{a} . \underline{b} \; x \; \underline{c} = \underline{a} . (\underline{b} \; x \; \underline{c})$

-This is sometimes called scalar triple product

Note

-If scalar triple product (box product) of three vectors $\underline{a}$. $\underline{b}$ and $\underline{c}$ = 0

-Then the vector $\underline{a}$, $\underline{b}$ and $\underline{c}$ are said to be COMPLANAR

Question

25. If $\underline{a} = 2\underline{i} + \underline{j} + 2\underline{k}$

$\underline{b} = 2\underline{i} + \underline{k}$ and

$\underline{c} = 3\underline{i} + 2\underline{j} + k$

Find

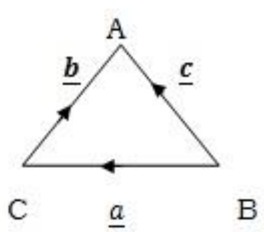
a) $\underline{a}.\underline{b} \times c$

b) $\underline{a} \times \underline{b}.\underline{c}$

APPLICATION OF CROSS PRODUCT

USED TO PROVE SINE RULE

- Consider the diagram below



$\underline{a} + \underline{b} + -\underline{c} = 0$

$\underline{a} + \underline{b} = \underline{c}$

Cross by $\underline{a}$ on both sides of

$\underline{a} \times \underline{b} + \underline{a} \times \underline{b} = \underline{a} \times \underline{c}$

$0 + \underline{a} \times \underline{b} = \underline{a} \times \underline{c}$

$\underline{a} \times \underline{a} + \underline{a} \times \underline{c}$

Crossing by $\underline{b}$ on both sides of eqn …. 1 above

$\underline{a} + \underline{b} = \underline{c}$

$\underline{b} \times \underline{a} + \underline{b} + \underline{b} = \underline{b} \times \underline{c}$

$\underline{b} \times \underline{a} + 0 = \underline{b} \times \underline{c}$

$\underline{b} \times \underline{a} = \underline{b} \times c$

$-(\underline{a} \times \underline{b)} = \underline{b} \times \underline{c}$

$\underline{a} \times \underline{b} = -\underline{(b} \times \underline{c})$

$\underline{a} \times \underline{b} = \underline{c} \times \underline{b}$

Equation i and ii as follows

$\underline{a} \times \underline{b} = \underline{a} \times \underline{c} = \underline{c} \times \underline{b}$

$|\underline{a}||\underline{b}| \text{ Sin } \hat{c} . \hat{n} = |\underline{a}||\underline{c}| \sin \hat{B} . \hat{n} = |\underline{c}||\underline{b}| \sin \hat{A} . \hat{n}$

$|\underline{a}||\underline{b}| \text{ Sin } \hat{c} = |\underline{a}||\underline{c}| \text{ Sin } \hat{B} = |\underline{c}||\underline{b}| \sin \hat{A}$

Dividing the whole eqn by

$|\underline{a}||\underline{b}||c|$

$\frac{|\underline{a}||\underline{b}|\sin \hat{c}}{|\underline{a}||\underline{b}||\underline{c}|} = \frac{|\underline{a}||\underline{c}|\sin \hat{B}}{|\underline{a}||\underline{b}||\underline{c}|} = \frac{|\underline{c}||\underline{b}|\sin \hat{A}}{|\underline{a}||\underline{b}||\underline{c}|}$

$\frac{\sin \hat{c}}{|\underline{c}|} = \frac{\sin \hat{n}}{|\underline{b}|} = \frac{\sin \hat{A}}{|\underline{a}|}$

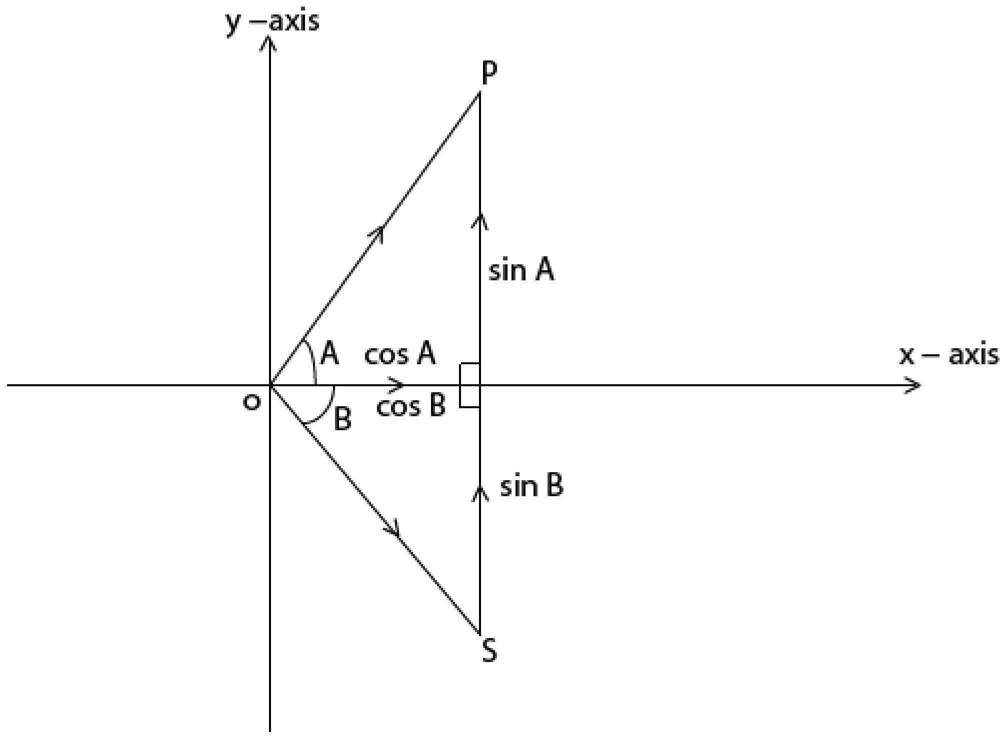
Sine rule

USED TO PROVED COMPOUND ANGLE FORMULA OF SINE

Sin (A + B) = sin A cos B + sin B cos A

Consider the vector diagram below

Pg. drawing



$\overrightarrow{op}$ = (cos A) $\underline{i}$ + (sin A) $\underline{j}$ + o$\underline{k}$

$\overrightarrow{os}$ = (cos A) $\underline{i}$ + (sin A) $\underline{j}$ + o$\underline{k}$

Hence

$|\overrightarrow{op}\ x\ \overrightarrow{os}|$ = $|\overrightarrow{op}||\overrightarrow{os}|$ sin (A + B)

But

$$\overrightarrow{op}x\ \overrightarrow{os} = \begin{bmatrix} i & j & k \\ \cos A & \sin A & 0 \\ \cos B & -\sin B & 0 \end{bmatrix}$$

$$\overrightarrow{op}x\ \overrightarrow{os} = i \begin{vmatrix} \sin A & 0 \\ -\sin B & 0 \end{vmatrix} + (-j) \begin{vmatrix} \cos A & 0 \\ \cos B & 0 \end{vmatrix} + k \begin{vmatrix} \cos A & \sin A \\ \cos B & -\sin B \end{vmatrix}$$

$\overrightarrow{op}x\ \overrightarrow{os}$ = i (o) – j (o) + k (-cos A sin B - sin A cos B)

$\overrightarrow{op} x \overrightarrow{os}$ = -k [sin A cos B + cos A sin B]

$|\overrightarrow{op} \; x \; \overrightarrow{os}| = \sqrt{[-(\sin A \cos B + \cos A \sin B)]_2}$

$= \sqrt{[(\sin A \cos B + \cos A \sin B)]_2}$

$|\overrightarrow{op} \; x \; \overrightarrow{os}|$ = sin A cos B + cos A sin B

Also

$|\overrightarrow{op}| = \sqrt{(\cos A)^2 + (\sin A)^2 + 0^2}$

$|\overrightarrow{os}| = \sqrt{(\cos B)^2 + (-\sin B)^2 + (o)_2}$

$|\overrightarrow{os}|_2 = \sqrt{cos^2 B + sin_2} B$

= 1

Therefore

Sin A cos B + cos A sin B = (1) (1) sin (A+B)

Sin (A + B) = Sin A cos B + cos A sin B

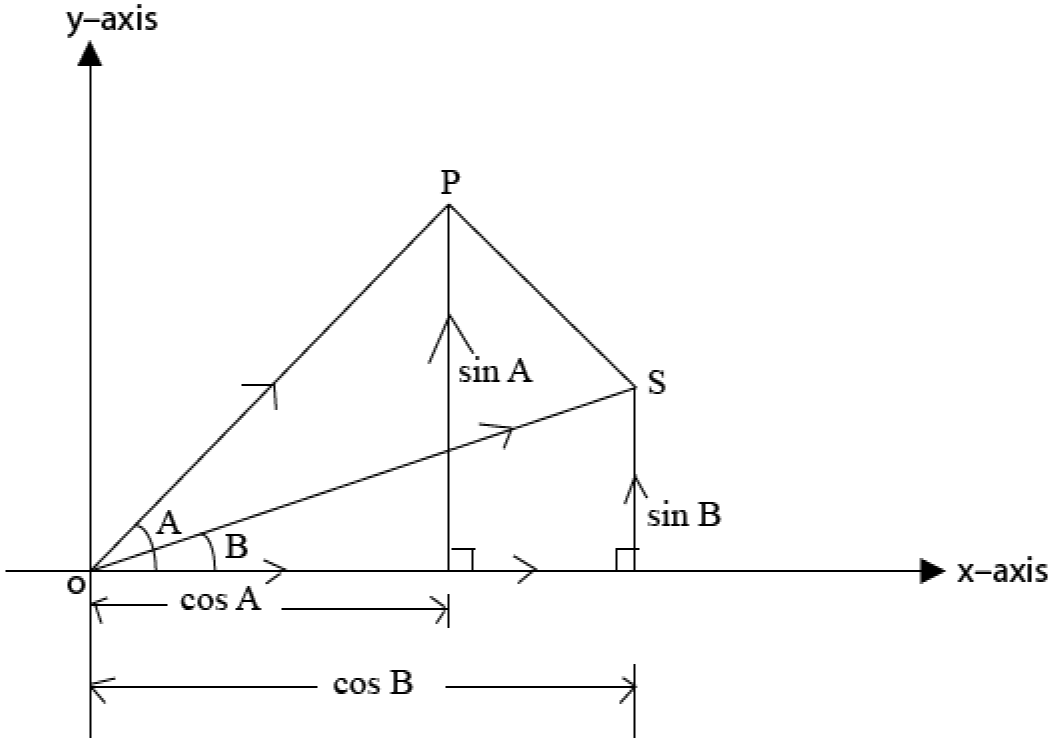
Proved

USED TO DETERMINE/ TO PROVE COMPOUND AND FORMULA OF SINE

Sin (A – B) = sin A cos B – Cos A sin B

- Consider the vector diagram below

Pg.6 drawing



$\overrightarrow{op}$ = (cos A) $\underline{i}$ + (sin A) $\underline{j}$ + o$\underline{k}$

$\overrightarrow{os}$ = (cos B) $\underline{i}$ + (sin B) $\underline{j}$ + o$\underline{k}$

= $|\overrightarrow{op} x \ \overrightarrow{os}|$ = $|\overrightarrow{op}||\overrightarrow{os}|$ sin (A - B)

but

$$\overrightarrow{op} x \ \overrightarrow{os} = \begin{bmatrix} i & j & k \\ \cos A & \sin A & 0 \\ \cos B & -\sin B & 0 \end{bmatrix}$$

$$= i \begin{vmatrix} \sin A & o \\ \sin B & 0 \end{vmatrix} - j \begin{vmatrix} \cos A & o \\ \cos B & 0 \end{vmatrix} + k \begin{vmatrix} \cos A & \sin A \\ \cos B & \sin B \end{vmatrix}$$

$|\overrightarrow{op}\ x\ \overrightarrow{os}|$ = $\sqrt{[-(\sin A \cos B - \cos A \sin B)]_2}$

= $\sqrt{[(\sin A \cos B - \cos A \sin B)]_2}$

= sin A cos B - cos A sin B

Also

$|\overrightarrow{op}|$ = $\sqrt{(\cos A)^2 + (\sin A)^2}$ + 0

= $\sqrt{cos\ ^2A + sin^2A}$

= $\sqrt{1}$

= 1

$|\overrightarrow{os}|$ = $\sqrt{(\cos B)^2 + (-\sin B)^2}$

= $\sqrt{cos\ ^2B + sin^2B}$

= $\sqrt{1}$

= 1

Therefore

Sin A cos B – cosA sin B = (1) (1) sin (A – B)

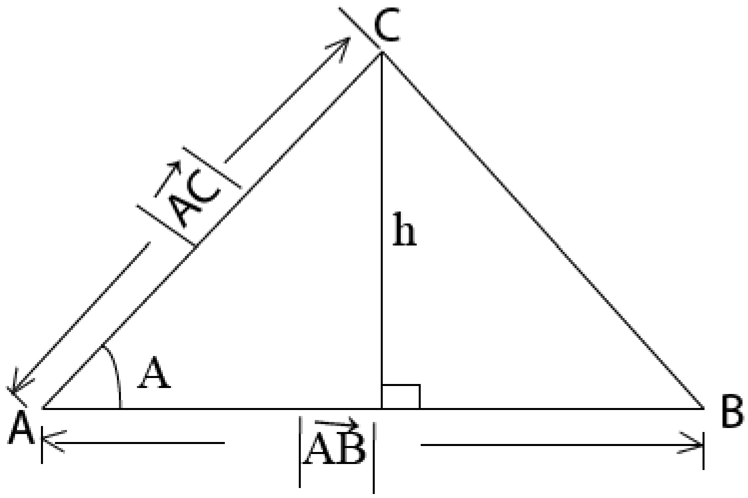
Sin (A – B) = sin A cos B – cos A sin B

Proved

USED TO FIND THE AREA OF THE TRIANGLE

- Consider the triangle ABC below

Pg. 7 drawing



Area (A) = ½ x base (b) x height (h)

A = ½ bh ……i

Also

Sin $\hat{A} = \frac{opp}{Hyp}$

Sin $\hat{A} = \frac{h}{|\vec{Ac}|}$

h = $|\vec{Ac}|$ Sin $\hat{A}$ ….ii

And

b = $\vec{AB}$ …iii

Substitute …ii and …iii into 1 as follows

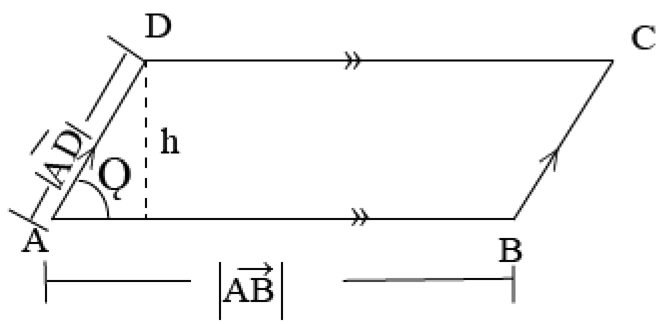
A = ½ $|\overrightarrow{AB}||\overrightarrow{AC}|$ sin $\hat{A}$

A = ½ $|\overrightarrow{AB}\ x\ \overrightarrow{AC}|$

USED TO FIND THE AREA OF PARALLELOGRAM

- Consider the parallelogram below

Pg. 7 drawing



Area = length (l) x height (h)

= A = Lh …i

L = $|\overrightarrow{AB}|$…ii

Also

Sin Q = $\frac{Opp}{hyp}$

$$\sin Q = \frac{h}{|\overrightarrow{AD}|}$$

h = $|\overrightarrow{AD}|$ Sin Q....ii

Substitute ii and iii into ---1 as follows

Area (A) = $|\overrightarrow{AB}||\overrightarrow{AD}|$ sin Q

A = $|\overrightarrow{AB}\ x\ \overrightarrow{AD}|$

Generally

Area (A) = $|\overrightarrow{AB}\ x\ \overrightarrow{BC}|$

= $|\overrightarrow{BC}\ x\ \overrightarrow{CD}|$

= $|\overrightarrow{CD}\ x\ \overrightarrow{DA}|$

= $|\overrightarrow{DA}\ x\ \overrightarrow{AB}|$

Where

$|\overrightarrow{AB}|$ = $\underline{b}$ - $\underline{a}$

$|\overrightarrow{BC}|$ = $\underline{c}$ - $\underline{a}$

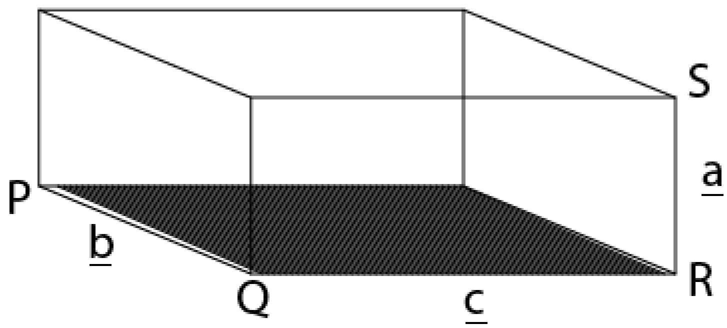
$|\overrightarrow{AC}|$ = $\underline{c}$ - $\underline{a}$

$|\overrightarrow{CD}|$ = $\underline{d}$ - $\underline{c}$

USED TO FIND THE VOLUME OF PARALLELOPIPED

- Consider the diagram below

Pg. 7 drawing



- Suppose P Q R and S are the vertices of the parallelepiped, hence the volume (v) of the parallelepiped is given by:

Volume (v) = base area (A) x height (h)

V = $|\overrightarrow{PQ} \times \overrightarrow{QR}| \times |\overrightarrow{RS}|$

V = $|\overrightarrow{PQ} \times \overrightarrow{QR} \cdot \overrightarrow{RS}|$

V = $|\overrightarrow{PQ} \times \overrightarrow{QR} \cdot \overrightarrow{RS}|$

V = $|\overrightarrow{PQ} \times \overrightarrow{QR} \cdot \overrightarrow{RS}|$

Again, for the sides with position vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$

Volume (v) = $|\underline{b} \times \underline{c}| \times |\underline{a}|$

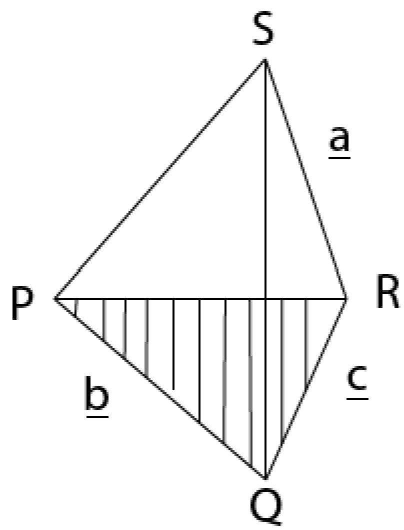
= $|\underline{b} \times \underline{c} \cdot \underline{a}|$

= $|\underline{a} \cdot \underline{b} \times \underline{c}|$

USED TO FIND THE VOLUME OF A TETRAHEDRON

- Consider the tetrahedron with vertices P, Q, R and S

Pg. 8 drawing



Volume (v) = 1/3 x base area x altitude

V = 1/3 x ½ $|\overrightarrow{PQ}\ x\ \overrightarrow{QR}|$ x $|\overrightarrow{RS}|$

V = 1/6 $|\overrightarrow{PQ}\ x\ \overrightarrow{QR}|$ x $|\overrightarrow{RS}|$

V = $|\overrightarrow{PQ}\ x\ \overrightarrow{QR}\ .\ \overrightarrow{RS}|$

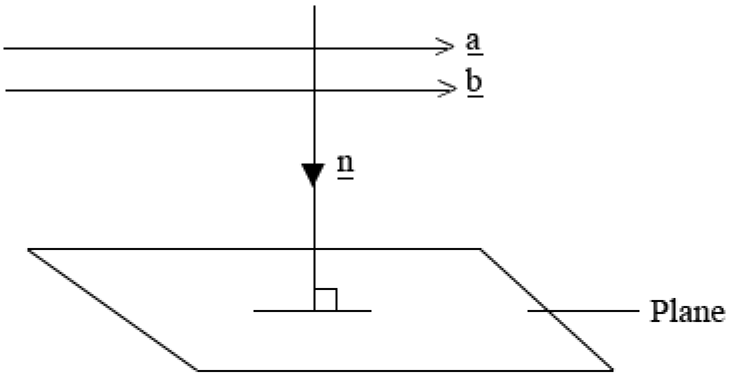
Therefore

Volume = 1/6 $|\underline{a} \cdot \underline{b} \times \underline{c}|$

USED TO FIND THE VECTOR PERPENDICULAR TO THE PLANE

- Consider the diagram below

Pg. 8 drawing



Where

$\underline{n}$ = is the vector perpendicular or normal to the plane

hence

$\underline{n} = \underline{a} \times \underline{b}$

Question

26. Find the area of the triangle ABC whose vertices are A (2, 1, 1) B (3, 2, 1) and C (-2, -4, -1)

27. The position vector of the points A, B and C are (2, 4, 3), (6, 3, -4) and (7, 5, -5) respectively. Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and hence the area of the triangle ABC

28. Find the area of the parallelogram whose vertices P, Q and R are (2, 1, 1), (3, 2, 1) (2, 4, 1)

29. Find the volume of the parallelepiped the edges are A (1, 0, 2) B (2, -1, 3) C (4, 1, 3) and D (1, -1, 1)

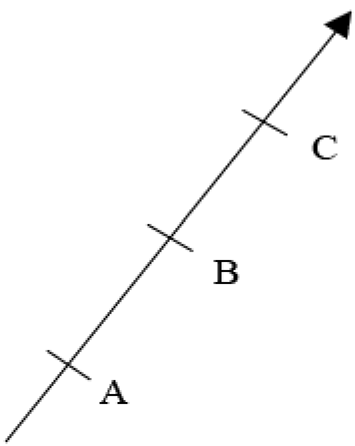
30. Find the volume of the tetrahedron whose sides are $\underline{a} = 2\underline{i} + \underline{k}$, $\underline{b} = \underline{i} - 3\underline{j} + \underline{k}$, and $\underline{c} = 3\underline{i} - \underline{j} + 2\underline{k}$

COLLINEAR AND COPLANAR VECTORS

1. **COLLINEAR VECTOR**

These are vectors having the same slope (re direction).

Pg. 10 drawing



$\overrightarrow{AB} = \propto \overrightarrow{AC}$

$\overrightarrow{AC} = \mu \overrightarrow{AC}$

$\overrightarrow{AB} = t \overrightarrow{BC}$

Where

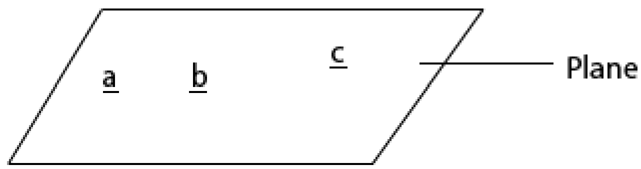
Æ›, µ and t are scalar

Again $\underline{a}$ and $\underline{b}$ to be collinear $\underline{a}$ x $\underline{b}$ = 0

2. **COPLANAR VECTOR**

These are vectors which lie on the same plane

Eg. **Pg. 10 drawing**



For the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ to be coplanar

$\underline{a} \cdot \underline{b} \times \underline{c} = 0$

$\underline{b} \cdot \underline{a} \times \underline{c} = 0$

$\underline{c} \cdot \underline{a} \times \underline{b} = 0$

Generally

$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{a} \times \underline{c} = \underline{c} \cdot \underline{a} \times \underline{b} = 0$

Question

31. Given that

$\underline{a} = 3\underline{i} + 4\underline{j}$

$\underline{b} = \propto i \text{ and}$

$\underline{c} = \underline{i} - 2\underline{j}$. Find $\propto$ if $\underline{a}, \underline{b}$, and $\underline{c}$ are collinear

32. Find the value are collinear vectors $2\underline{i} - \underline{j} + \underline{k}$, $\underline{i} + 2\underline{j} + 3\underline{k}$ and $3\underline{i} + \underline{\lambda j} + 5\underline{k}$ are coplanar.

33. Find unit vector in the direction of $\underline{a} = 6\underline{i} + 3\underline{j} + \underline{k}$ and state its length

LINEAR COMBINATION OF VECTORS

Suppose that $\underline{a}$, $\underline{b}$ and $\underline{c}$ are vectors and α, β and γ are real numbers (sealers). Then a vector $\underline{k}$ = $\propto \underline{a} + \beta \underline{b} + \gamma \underline{c}$ is a linear combination of vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$

NB

To solve vectors means to put the vectors into linear form

Question

34. If $\underline{a} = \underline{i} + \underline{j}$, $\underline{b} = \underline{i} - \underline{j}$ and $\underline{c} = 3\underline{i} - 4\underline{j}$ resolve $\underline{c}$ into vectors parallel to $\underline{a}$ and $\underline{b}$

35. Express the vector r = $10\underline{i} - 3\underline{j} - \underline{k}$ as a linear function of $\underline{a}$, $\underline{b}$ and $\underline{c}$ given that

$\underline{a} = 2\underline{i} - \underline{j} + \underline{3k}$

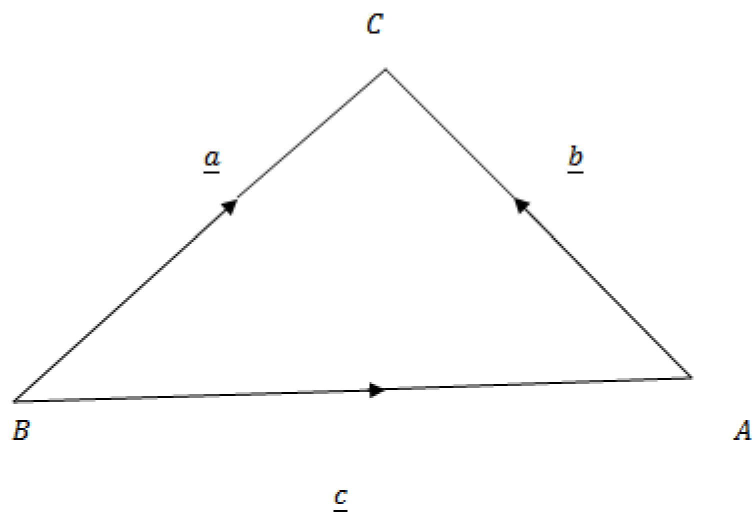
$\underline{b} = 3\underline{i} + 2\underline{j} - \underline{4k}$ and

$\underline{c} = -\underline{i} + 3\underline{j} - 2\underline{k}$

Note: Required to be placed in a right position

Subtopic: Dot Product

Proving cosine rule using dot product



Consider the triangle ABC above

$\underline{a} + \underline{b} = \underline{c}$

$\underline{a} = \underline{c} - \underline{b}$

$$Suppose$$

$$\underline{a} \,.\, \underline{a} = [\underline{c} - \underline{b}][\underline{c} - \underline{b}]$$

$$|\underline{a}|^2 = |\underline{c}|^2 - |2\underline{bc}| + |\underline{b}|^2$$

$$|\underline{a}|^2 = |\underline{c}|^2 - |2\underline{bc}| + |\underline{b}|^2$$

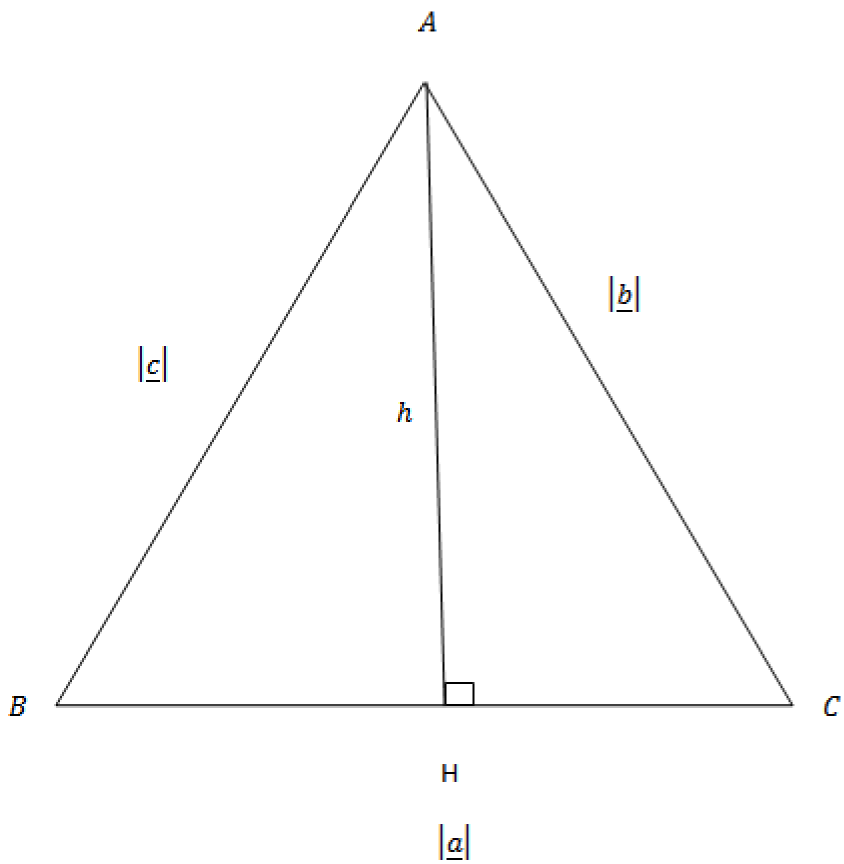
$$but$$

$$\underline{b} \,.\, \underline{c} = |\underline{b}||\underline{c}| \cos A$$

$$|\underline{a}|^2 = |\underline{c}|^2 + |\underline{b}|^2 - 2\underline{bc} \cos A \,.\; Hence\ proved$$

Subtopic: Cross Product

Proof of sine rule Consider the â^†ABC with sides A,B and C respectively



Construct a line AH which lies on BC and perpendicular.

$$\sin B = \frac{H}{|c|}$$

$$h = |c| \sin B$$

$$Similary$$

$$\sin B = \frac{H}{|b|}$$

$$h = |b| \sin C$$

$$Area\ of\ triangle\ ABC$$

$$A = \frac{1}{2} bh$$

$$A = \frac{1}{2} |\underline{b}||\underline{a}| \sin C$$

$$A = \frac{1}{2} |\underline{a}||\underline{c}| \sin B$$

$$A = \frac{1}{2} |\underline{c}||\underline{b}| \sin A$$

Equating area

$$\frac{1}{2} |\underline{b}||\underline{a}| \sin C = \frac{1}{2} |\underline{a}||\underline{c}| \sin B = \frac{1}{2} |\underline{c}||\underline{b}| \sin A$$

$$Multipling\ by\ 2\ throughout$$

$$|\underline{b}||\underline{a}| \sin C = |\underline{a}||\underline{c}| \sin B = |\underline{c}||\underline{b}| \sin A$$

$$Dividing\ by\ |\underline{a}||\underline{b}||\underline{c}|\ throughout$$

$$\frac{|b||\underline{a}| \sin C}{|\underline{a}||\underline{b}||\underline{c}|} = \frac{|\underline{a}||\underline{c}| \sin B}{|\underline{a}||\underline{b}||\underline{c}|} = \frac{|\underline{c}||\underline{b}| \sin A}{|\underline{a}||\underline{b}||\underline{c}|}$$

$$\frac{\sin C}{|\underline{c}|} = \frac{\sin B}{|\underline{b}|} = \frac{\sin A}{|\underline{a}|}\ Hence\ the\ sine\ rule\ proved$$

HYPERBOLIC FUNCTION

Hyperbolic of sine - Sinh x.

Hyperbolic of cosine - coshnx

Hyperbolic of tangents - tanhnx

Hyperbolic of cotangents - cothnx

Hyperbolic of secants - Sechnx

Hyperbolic of cosecants - cosechnx

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$cosech\ x = \frac{1}{\sinh x}$$

$from$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots (i)$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots (ii)$$

$adding\ equation\ (i) and\ (ii)$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots$$

$$\frac{e^x + e^{-x}}{2} = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

but

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

upon subtracting equation (i) *and* (ii)

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$but \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$= \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$= \frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2}}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\operatorname{sech} x = \frac{1}{\frac{e^x + e^{-x}}{2}}$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$cosech\ x = \frac{1}{\sinh x}$$

$$cosech\ x = \frac{1}{\frac{e^x - e^{-x}}{2}}$$

$$cosech\ x = \frac{2}{e^x - e^{-x}}$$

FURTHER IDENTITIES

- $\cosh x + \sinh x = \frac{e^x+e^{-x}}{2} + \frac{e^x-e^{-x}}{2} = \frac{1}{2}e^x + \frac{e^{-x}}{2} + \frac{e^x}{2} - \frac{e^{-x}}{2}$

$$\rightarrow \cosh x + \sinh x = e^x \ \ldots\ldots (i)$$

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}e^x + \frac{1}{2}e^{-x} - \frac{1}{2}e^x + \frac{1}{2}e^{-x}$$

$$\rightarrow \cosh x - \sinh x = e^{-x} \ \ldots\ldots (ii)$$

multiplying equation (i) and (ii)

$$(\cosh x + \sinh x)(\cosh x - \sinh x) = e^x e^{-x}$$

$$\rightarrow cosh^2 x - sinh^2 x = 1 \ \ldots\ldots\ldots\ldots (iii)$$

divide equation (iii) above by $cosh^2 x$ *through out*

$$cosh^2 x - sinh^2 x = 1$$

$$\frac{cosh^2 x}{cosh^2 x} - \left(\frac{sinh^2 x}{cosh^2 x}\right) = \frac{1}{cosh^2 x}$$

$$\rightarrow 1 - tanh^2 x = sech^2 x$$

divide equation (iii) above by $sinh^2 x$

$$\frac{cosh^2 x}{sinh^2 x} - \left(\frac{sinh^2 x}{sinh^2 x}\right)$$

$$\rightarrow coth^2 x - 1 = cosech^2 x$$

then take

$$\begin{cases} \cosh x + \sinh x = e^{x} \ldots\ldots(i) \\ \cosh x - \sinh x = e^{-x} \ldots..(ii) \end{cases}$$

$$(\cosh x + \sinh x)^2 = (e^x)^2$$

$$cosh^2 x + 2\cosh x \sinh x + sinh^2 x = e^{2x} \ldots\ldots\ldots\ldots\ldots(*)$$

$$(\cosh x - \sinh x)^2 = (e^{-x})^2$$

$$cosh^2 x - 2\cosh x \sinh x + sinh^2 x = e^{-2x} \ldots\ldots\ldots\ldots\ldots(**)$$

$add\ eequation * and **$

$$+\begin{cases} cosh^2 x + 2\cosh x \sinh x + sinh^2 x = e^{2x} \\ cosh^2 x - 2\cosh x \sinh x + sinh^2 x = e^{-2x} \end{cases}$$

$$2cosh^2 x + 2sinh^2 x = \frac{e^{2x} + e^{-2x}}{2}$$

$$cosh^2 x + sinh^2 x = \cosh 2x$$

$from$

$$cosh^2 x + sinh^2 x = 1$$

$$Cosh^2 x = 1 - Sinh^2 x$$

$then$

$$1 + sinh^2 x + sinh^2 x = \cosh 2x$$

$$1 + 2sinh^2 x = \cosh 2x \rightarrow \cosh 2x = 1 + 2sinh^2 x$$

Or from

$$cosh^2 x - sinh^2 x = 1$$

$$sinh^2 x = cosh^2 x - 1$$

$then$

$$cosh^2 x + cosh^2 x - 1 = \cosh 2x$$

$$\cosh 2x = 2cosh^2 x - 1$$

$subtracting\ equations$

$$-\begin{cases} cosh^2 x + 2\cosh x \sinh x + sinh^2 x = e^{2x} \\ cosh^2 x - 2\cosh x \sinh x + sinh^2 x = e^{-2x} \end{cases}$$

$$4CoshxSinhx = e^{2x} - e^{-2x}$$

$$2(2\cosh x \times \sinh x) = e^{2x} - e^{-2x}$$

$$2\cosh x \sinh x = \frac{e^{2x} - e^{-2x}}{2}$$

$$2\cosh x \sinh x = \sinh 2x$$

$$\rightarrow \mathbf{\sinh 2x = 2\cosh x \sinh x}$$

$$\tanh 2x = \frac{\sinh 2x}{\cosh 2x}$$

$$\tanh 2x = \frac{2\cosh x \sinh x}{cosh^2 x + sinh^2 x}$$

$divide\ by\ cosh^2 x\ through\ out$

$$\tanh 2x = \frac{\frac{2\cosh x \sinh x}{cosh^2 x}}{\frac{cosh^2 x}{cosh^2 x} + \frac{sinh^2 x}{cosh^2 x}}$$

$$= \frac{\frac{2\sinh x}{\cosh x}}{1 + \left(\frac{\sinh x}{\cosh x}\right)^2}$$

$$\rightarrow \tanh 2x = \frac{2\tanh x}{1 + tanh^2 x}$$

<u>T- FORMULAE</u>

(a) $\tanh 2x = \frac{2\tanh x}{1+tanh^2 x} divide\ by\ 2\ through\ out$

$$\tanh x = \frac{2\tanh\frac{x}{2}}{1 + tanh^2\frac{x}{2}}$$

$let\ \tanh x = t$

$$\rightarrow \mathbf{\tanh x = \frac{2t}{1 + t^2}}$$

(b) $\sinh 2x \sinh 2x = \frac{2\sinh x\cosh x}{1}$

$= \frac{2\sinh x\cosh x}{cosh^2 x - sinh^2 x}$

$divide\ by\ cosh^2 x$

$\sinh 2x = \frac{\frac{2\sinh x\cosh x}{cosh^2 x}}{\frac{cosh^2 x}{cosh^2 x} - \frac{sinh^2 x}{cosh^2 x}}$

$\sinh 2x = \frac{2\tanh x}{1 - tanh^2 x}$

$divide\ by\ 2$

$\sinh x = \frac{2\tanh\frac{x}{2}}{1 - tanh^2\frac{x}{2}}$

$\rightarrow \mathbf{\sinh x = \frac{2t}{1 - t^2}}$

(c) $\cosh 2x \cosh 2x = \frac{cosh^2 x + sinh^2 x}{1}$

$\cosh 2x = \frac{cosh^2 x + sinh^2 x}{cosh^2 x - sinh^2 x}$

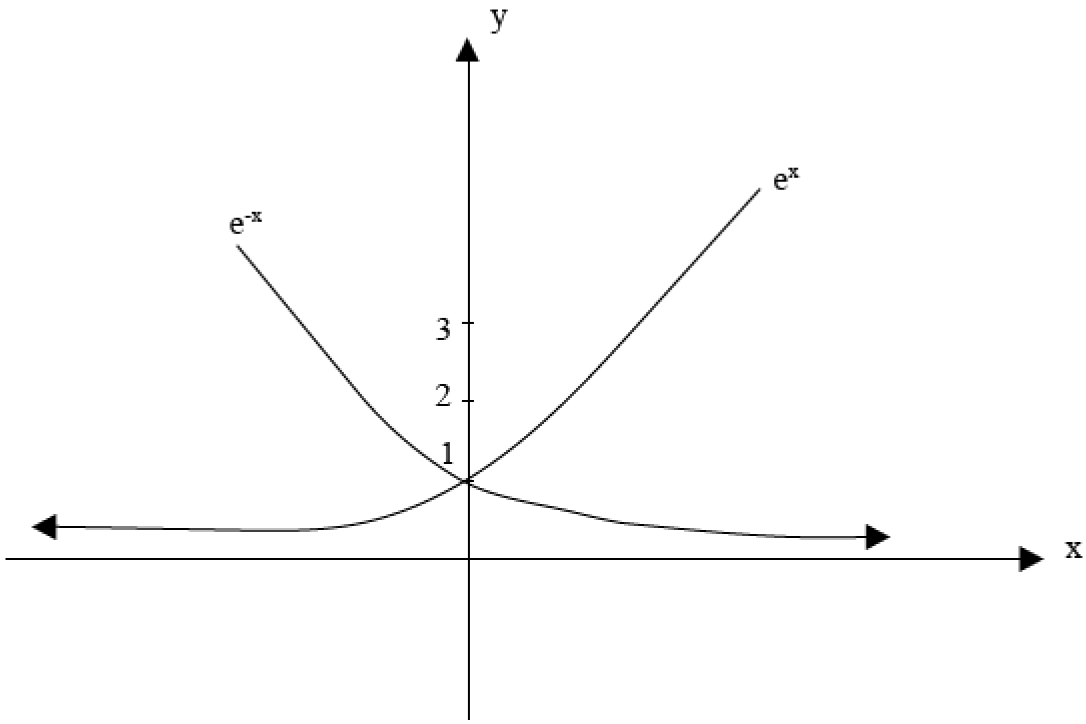
$\cosh 2x = \frac{1 + tanh^2 x}{1 - tanh^2 x}$

$divide\ by\ 2$

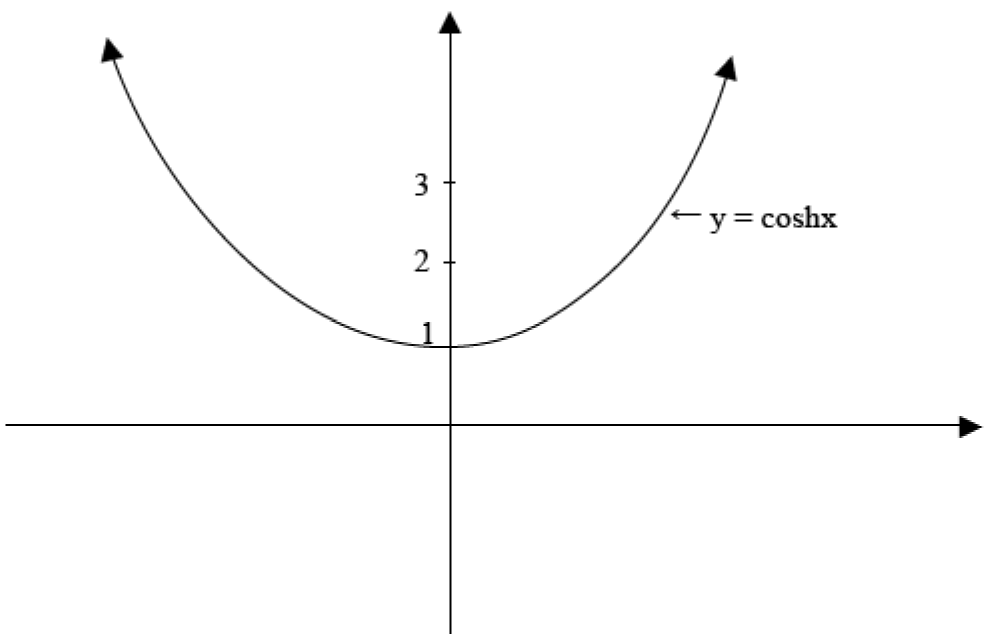
$\cosh x = \frac{1 + tanh^2\frac{x}{2}}{1 - tanh^2\frac{x}{2}}$

$\rightarrow \mathbf{\cosh x = \frac{1 + t^2}{1 - t^2}}$

GRAPHS OF HYPERBOLIC FUNCTION



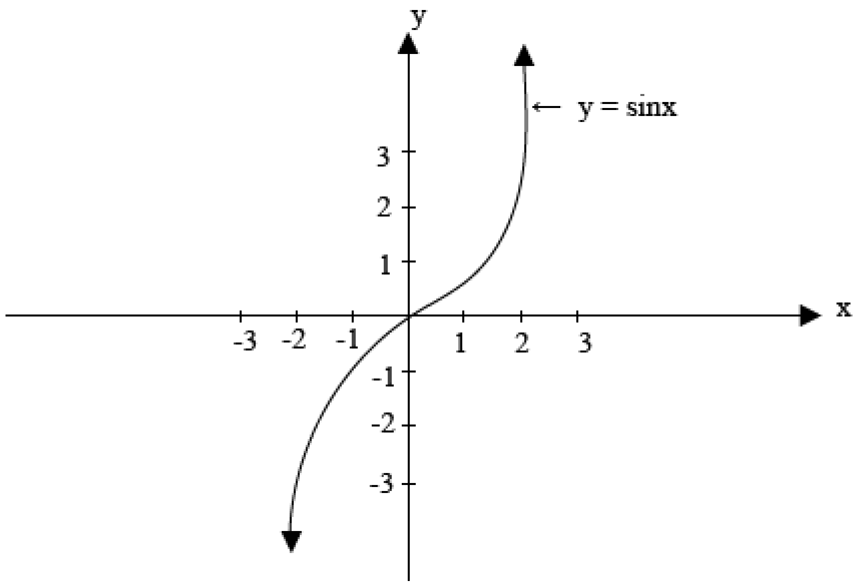
_(1)Y=cosh x



Domain = $\{x: x \text{ is all real number}\}$

Range = $\{y: y \geq 1\}$

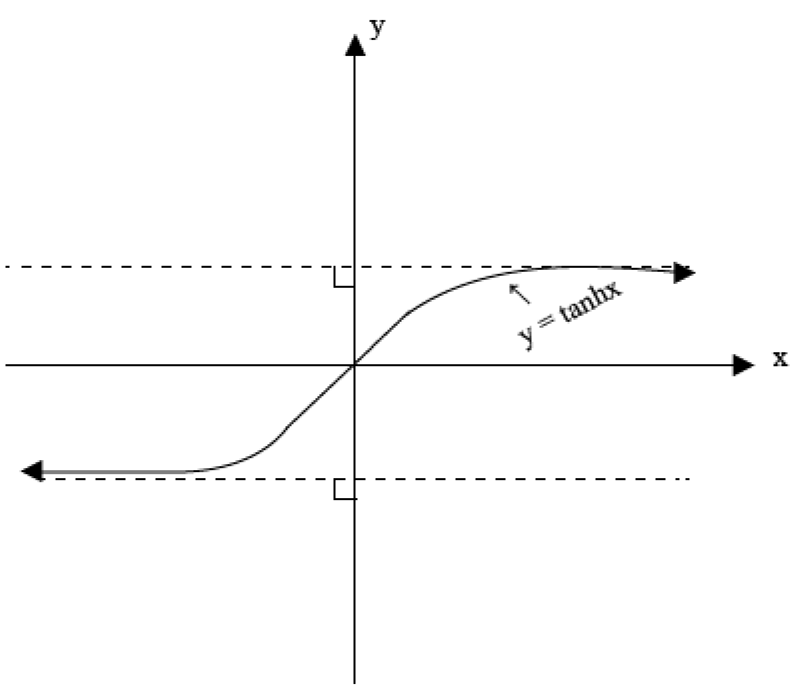
(ii) y = sinhx



$domain = \{x : x\ is\ all\ real\ numbers\}$

$range = \{y : y\ is\ all\ real\ numbers\}$

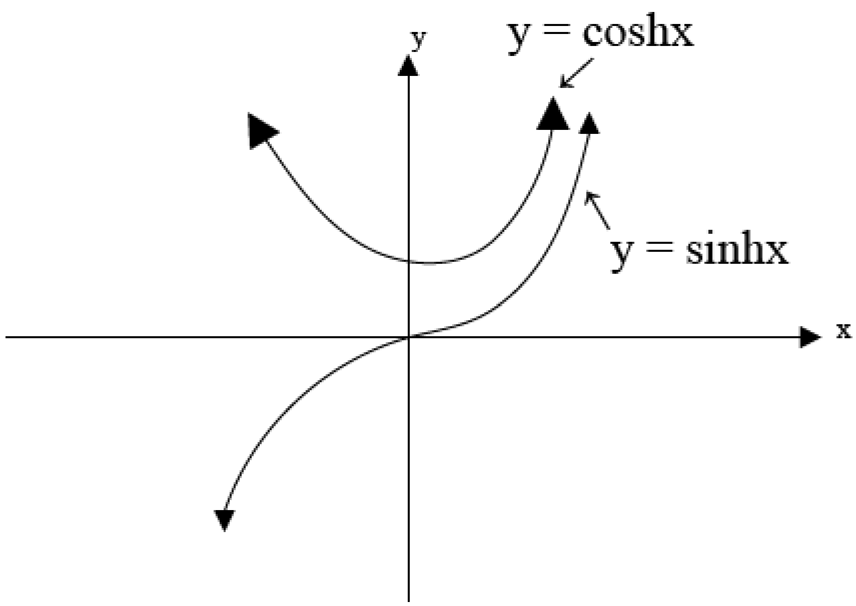
(iii) y = tanhx



$domain = \{x : x \text{ is all real numbers}\}$

$range = \{y : y, -1 < y < 1\}$

COMBINATION OF COSHX AND SINHX



Examples

- 1. *Show that*

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

- 2. *Show that*

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

- 3. *Show that*

$$\frac{1 + \tanh x}{1 - \tanh x} = e^{2x}$$

- 4. *Show that*

$$\tanh(A+B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$$

- 5. Find the expression that equation $a\cosh x + b\cosh x = c$ has equal roots

- If $e^u = (\sec x + \tan x)$ Prove that $\cosh u = \sec x$

- 6. Show that the point $(a\cosh x, b\cosh x)$ lies between $\frac{x^2}{a^2} - \frac{y^2}{b^2}$

- 7. Solve the following

  (i) $3\sinh x - \cosh x = 1$

- 8.Solve the following equation $3sech^2 x + 4\tanh x + 1 = 0$ give the roots in terms of natural logarithms

- if $\sinh u = \tanh\theta,\ show\ that\ U = \ln(\sec\theta + \tan\theta)$

<u>Solution 01</u>

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$take\ R.H.S$

$\sinh x \cosh y + \cosh x \sinh y$

$but \sinh x = \frac{e^x - e^{-x}}{2}$

$\sinh y = \frac{e^y - e^{-y}}{2}$

$\cosh y = \frac{e^y + e^{-y}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$$\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$\left(\frac{1}{2}\times\frac{1}{2}\right)[e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}] + \left(\frac{1}{2}\times\frac{1}{2}\right)[e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}]$$

$$= \frac{1}{4}[e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y} + e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}]$$

$$= \frac{1}{4}[2e^{x+y} - 2e^{-x-y}]$$

$$= \frac{1}{4}\times 2\left[e^{x+y} - e^{-(x+y)}\right]$$

$$= \frac{1}{2}\left[e^{x+y} - e^{-(x+y)}\right]$$

$$\frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$$

$$\therefore \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

<u>Solution 2</u>

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$take\ R.H.S$

$$\cosh x \cosh y + \sinh x \sinh y$$

$$\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \left(\frac{1}{2}\times\frac{1}{2}\right)[e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}] + \left(\frac{1}{2}\times\frac{1}{2}\right)[e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}]$$

$$= \frac{1}{4}[e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y} + e^{x+y} + e^{x-y} - e^{-x+y} + e^{-x-y}]$$

$$= \frac{1}{4}(2e^{x+y} + 2e^{-x-y})$$

$$= \frac{1}{4}\times 2\left[e^{x+y} + e^{-(x+y)}\right]$$

$$= \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x+y)$$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y \qquad \textit{hence shown}$

Solution 03

$\frac{1+\tanh x}{1-\tanh x} = e^{2x}$ *take L.H.S*

$$\frac{1+\tanh x}{1-\tanh x}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \frac{1 + \frac{e^x - e^{-x}}{e^x + e^{-x}}}{1 - \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]}$$

$$= \frac{\frac{1(e^x + e^{-x}) + e^x - e^{-x}}{e^x + e^{-x}}}{\frac{e^x + e^{-x} - 1(e^x - e^{-x})}{e^x + e^{-x}}}$$

$$= \frac{e^x + e^{-x} + e^x - e^{-x}}{e^x + e^{-x} - e^x + e^{-x}}$$

$$= \frac{2e^x}{2e^{-x}}$$

$$= e^x . e^x$$

$$= e^{2x}$$

$$\therefore \frac{1+\tanh x}{1-\tanh x} = e^{2x} \qquad \textit{shown}$$

Solution 04

$$\tanh(A+B)=\frac{\tanh A+\tanh B}{1+\tanh A\tanh B}$$

take R.H.S

$$tanh\ (A+B)=\frac{tanhA+tanhB}{1+tanhAtanhB}$$

$$tanhA=\frac{e^{A}-e^{-A}}{e^{A}+e^{-A}}$$

$$tanhB=\frac{e^{B}-e^{-B}}{e^{B}+e^{-B}}$$

$$=\frac{tanhA+tanhB}{1+tanhAtanhB}$$

$$=\frac{\frac{e^{A}-e^{-A}}{e^{A}+e^{-A}}+\frac{e^{B}-e^{-B}}{e^{B}+e^{-B}}}{1+(\frac{A-e^{-A}}{e^{A}+e^{-A}})(\frac{e^{B}-e^{-B}}{e^{B}+e^{-B}})}$$

$$= \frac{\frac{(e^{A}-e^{-A})(e^{B}+e^{-B})+(e^{B}-e^{-B})(e^{A}+e^{-A})}{(e^{A}+e^{-A})(e^{B}+e^{-B})}}{\frac{(e^{A}+e^{-A})(e^{B}+e^{-B})+(e^{A}-e^{-A})(e^{B}-e^{-B})}{(e^{A}+e^{-A})(e^{B}+e^{-B})}}$$

$$= \frac{(e^{A}-e^{-A})(e^{B}+e^{-B})+(e^{B}-e^{-B})(e^{A}+e^{-A})}{(e^{A}+e^{-A})(e^{B}+e^{-B})} \times \frac{(e^{A}+e^{-A})(e^{B}+e^{-B})}{(e^{A}+e^{-A})(e^{B}+e^{-B})+(e^{A}-e^{-A})(e^{B}-e^{-B})}$$

$$= \frac{(e^{A}-e^{-A})(e^{B}+e^{-B})+(e^{B}-e^{-B})(e^{A}+e^{-A})}{(e^{A}+e^{-A})(e^{B}+e^{-B})+(e^{A}-e^{-A})(e^{B}-e^{-B})}$$

$$= \frac{2e^{A+B} - 2e^{-A-B}}{2e^{A+B} + 2e^{-A-B}}$$

$$= \frac{2e^{A+B} - 2e^{-(A+B)}}{2e^{A+B} + 2e^{-(A+B)}}$$

$$= \frac{2(e^{A+B} - e^{-(A+B)})}{2(e^{A+B} + e^{-(A+B)})}$$

$$= \frac{e^{A+B} - e^{-(A+B)}}{e^{A+B} + e^{-(A+B)}}$$

$But\ \frac{e^{A+B} - e^{-(A+B)}}{e^{A+B} + e^{-(A+B)}} = \tanh(A+B)$ $\qquad HENCE\ SHOWN$

Solution 05

$a\cosh x + b\sin x = c$

$a\left[\frac{1+t^2}{1-t^2}\right] + b\left[\frac{2t}{1-t^2}\right] = c$

$\frac{a(1+t^2)+b(2t)}{1-t^2} = c$

$a(1+t^2)+b(2t) = c(1-t^2)$

$a + at^2 + 2bt = c - ct^2$

$a + at^2 + 2bt - c + ct^2 = 0$

$at^2 + \ ct^2 + 2bt + a - c = 0$

$for\ two\ equal\ roots$

$b^2 = 4ac \quad where$

$a = a + c$

$b = 2b$

$c = a - c then, \ (2b)^2 = 4(a+c)(a-c)$

$4b^2 = 4(a^2 - c^2)$
$b^2 = a^2 - c^2$
$\therefore the\ condition\ is \quad b^2 = a^2 - c^2$

Solution 06

$\cosh u = \frac{e^u + e^{-u}}{2}$

$\cosh u = \frac{1}{2}\left[e^u + \frac{1}{e^u}\right]$

$\cosh u = \frac{1}{2}\left[(\sec x + \tan x) + \frac{1}{(\sec x + \tan x)}\right]$

$\cosh u = \frac{1}{2}\left[\frac{(\sec x + \tan x)(\sec x + \tan x) + 1}{(\sec x + \tan x)}\right]$

$\cosh u = \frac{1}{2}\left[\frac{sec^2 x + 2\sec x \tan x + tan^2 x + 1}{(\sec x + \tan x)}\right]$

$$\cosh u = \frac{1}{2}\left[\frac{sec^2 x + 2\sec x \tan x + sec^2 x}{(\sec x + \tan x)}\right]$$

$$\cosh u = \frac{1}{2}\left[\frac{2sec^2 x + 2\sec x \tan x}{(\sec x + \tan x)}\right]$$

$$\cosh u = \frac{1}{2}\left[\frac{2\sec x\,(\sec x + \tan x)}{\sec x \tan x}\right]$$

$$cush\ u = \frac{1}{2}(2)\sec x$$

$$\cosh u = \sec x$$

hence shown

Solution 07

$let\ \text{acosh}\, x = x$

$bsinh\ x = y$

$x^2 = a^2 cosh^2 x$

$y^2 = b^2 sinh^2 x$

substitute into equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{a^2 cosh^2 x}{a^2} - \frac{b^2 sinh^2 x}{b^2} = 1$$

$$cosh^2 x - sinh^2 x = 1$$

$$1 = 1$$

$since\ L.H.S = R.H.S\ hence\ (\text{acosh}\, x, bsinh\ x) lies\ on\ the\ hyperbola$

Solution 08

$$3\sinh x - \cosh x = 1$$

$$\frac{3(e^x - e^{-x})}{2} - \frac{(e^x + e^{-x})}{2} = 1$$

$$\frac{3e^x - 3e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = 1$$

$$\frac{3e^x - 3e^{-x} - e^x - e^{-x}}{2} = 1$$

$$3e^x - 3e^{-x} - e^x - e^{-x} = 2$$

$$2e^x - 4e^{-x} = 2$$

$$2(e^x - 2e^{-x}) = 2$$

$$e^x - 2e^{-x} = 1$$

Divide by e^{-x} throughout.

$$\frac{e^x}{e^{-x}} - \frac{2e^{-x}}{e^{-x}} = \frac{1}{e^{-x}}$$

$$e^x.\frac{1}{e^{-x}} - 2 = e^x$$

$$e^x.e^x - 2 = e^x$$

$$e^{2x} - e^x - 2 = 0$$

$$a = 1, b = -1, c = -2$$

$$e^x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

$$e^x = \frac{1 \pm \sqrt{1+8}}{2}$$

$$e^x = \frac{1 \pm 3}{2}$$

$$e^x = 2 \quad or \quad (-1)\ neglected$$

$$e^x = 2$$

$$apply\ ln$$

$$\ln(e^x) = \ln 2$$

$$x = \ln 2$$

$$x = 0.693$$

Solution 09

$3sech^2 x + 4\tanh x + 1 = 0$

but

$1 - tanh^2 x = sech^2 x$

$3(1 - tanh^2 x) + 4\tanh x + 1 = 0$

$3 - 3tanh^2 x + 4\tanh x + 1 = 0$

$-3tanh^2 x + 4\tanh x + 1 = 0$

$3tanh^2 x - 4\tanh x - 4 = 0$

$but\ by\ general\ solution$

$$\tanh x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-4)}}{2(3)}$$

$$\tanh x = \frac{4 \pm 8}{6}$$

$$\tanh x = \frac{4+8}{6} \quad or \quad \frac{4-8}{6}(neglected)$$

$$\mathrm{x} = \frac{1}{2}\ln\left(\frac{1+\frac{-2}{3}}{1-\frac{-2}{3}}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{1}{5}\right)$$

Solution 10

$\sinh u = \tan\theta$

$$\sinh u = \frac{e^u - e^{-u}}{2}$$

$$\frac{e^u - e^{-u}}{2} = \tan\theta$$

$e^u - e^{-u} = 2\tan\theta$

$(e^u - e^{-u})e^u = 2\tan\theta\, e^u$

$e^{2u} - 1 = 2e^u tan\theta$

$(e^u)^2 - (2\tan\theta)e^u - 1 = 0$

$$e^u = \frac{2tan\theta \pm \sqrt{4tan^2\theta + 4}}{2}$$

$u = \ln(sec\theta + tan\theta)$ *hence shown*

OSBORN'S RULE

This is the rule used to change the trigonometrical identification into corresponding analogous hyperbolic identities.

Osborn's rule states that "whenever a product of two series occurs change the sign of that term "

Examples

1. Change the identity $cos^2x + sin^2x = 1$ into analogous hyperbolic identity

Solution

$cos^2x + sin^2x = 1$

$\therefore cosh^2x - (sinh^2x) = 1$

2. Write the analogous hyperbolic identity for $\tan(A + B) = \frac{tanA + tanB}{1 - tanAtanB}$

Solution

$$tanh\ (A+B)=\frac{tanhA+tanhB}{1+tanhAtanhB}$$

then,

3. Change $\sin 2x = 2\sin x \cos x$ into a corresponding hyperbolic identity

Solution

$\sin 2x = 2sinxcosx$

$\sinh 2x = 2\sinh x \cosh x$

INVERSE OF HYPERBOLIC FUNCTION

The inverse of $\sinh x$ or $\text{arcsinh}\, x$

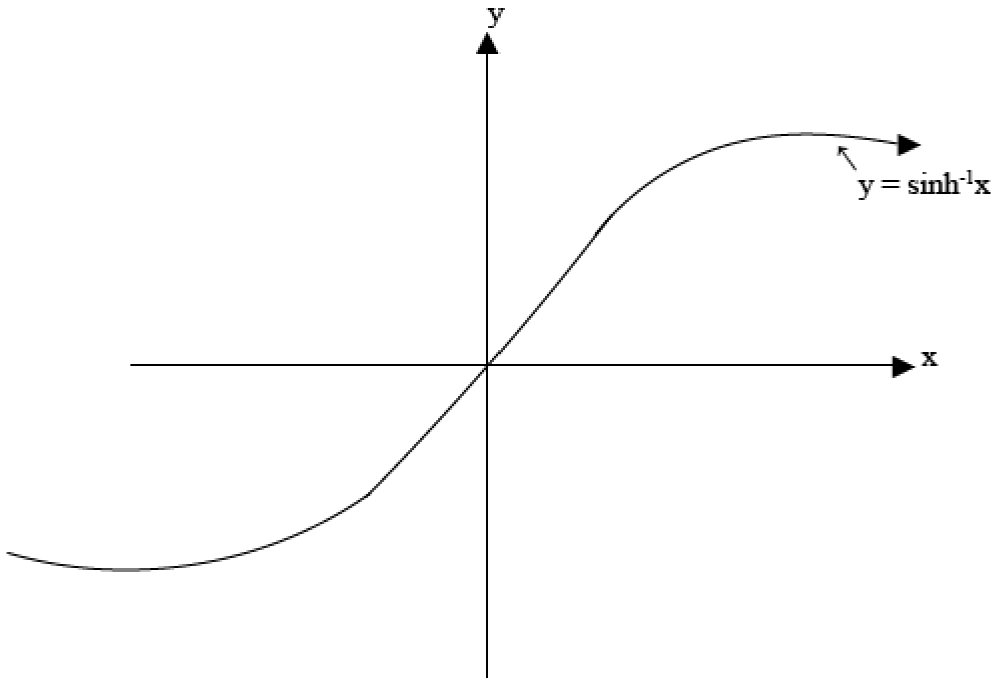
The inverse of $\cosh x$ is denoted by $\cosh^{-1} x \quad or\ \text{arccosh}\, x$

The inverse of $\tanh x$ is denoted by $\tanh^{-1} x \quad or\ \text{arctanh}\, x$

GRAPHS OF INVERSE OF HYPERBOLIC FUNCTIONS

The graph of the inverse of hyperbolic functions is a reflection of graphs of hyperbolic function on the inverse of y = x

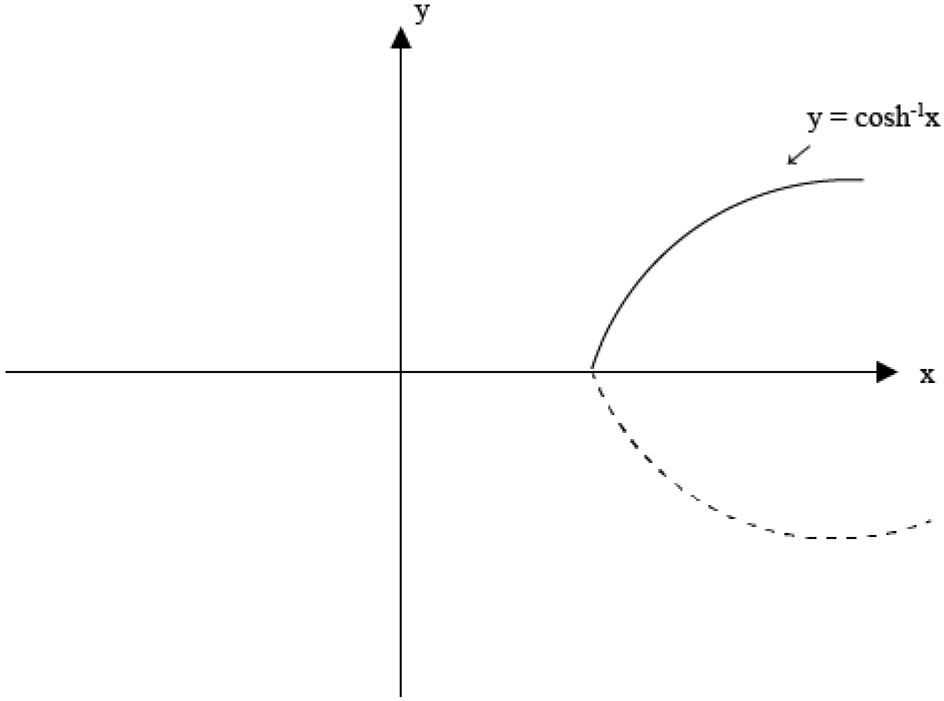
(a) $\sinh^{-1} x$



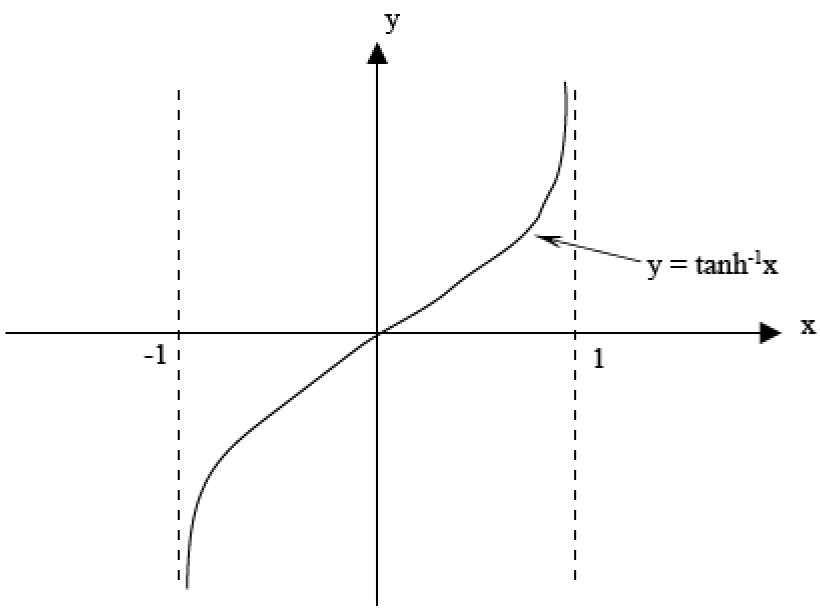
(b) $\cosh^{-1} x$

Concept:

y = $\cosh x$ is not one to one function in such a way it can't have inverse without restriction otherwise its inverse will not be a function but just a relation. For y = $\cosh^{-1} x$ to be a function the domain of y = $\cosh x$ should be restricted such that domain is $x \geq 0$



(c) $\tanh^{-1} x$



(d) For y = $\tanh^{-1} x$ it is defined by only for -1 < x < 1

EXPRESSION OF $sinh^{-1} x , cosh^{-1} x$ AND $tanh^{-1} x$ IN LOGARITHMIC FORM

(a) $y = \sinh^{-1} x$

$x = \sinh y$

$y = \frac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$e^y - 2x - e^{-y} = 0$

$multiply\ by\ \ e^y$

$e^{2y} - 2xe^y - 1 = 0$

$by\ general\ solution$

$e^y = \frac{2x \pm \sqrt{(2x)^2 - 4(1)(-1)}}{2(1)}$

$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$

$e^y = \frac{2x \pm 2\sqrt{x^2 + 1}}{2}$

$e^y = x \pm \sqrt{x^2 + 1}$

but

$e^y \geq 1$

$\therefore e^y = x + \sqrt{x^2 + 1}$

$applying \ln\ both\ sides$

$\ln e^y = \ln\left(x + \sqrt{x^2 + 1}\right)$

$y = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\therefore \sinh^{-1} x = \left(x + \sqrt{x^2 + 1}\right)$

(b b) $y = \cosh^{-1} x$

$\cosh y = x$

$\frac{e^y + e^{-y}}{2} = x$

$e^y + e^{-y} = 2x$

$e^y - 2x + e^{-y} = 0$

multiply by e^y *on both sides*

$e^y - 2xe^y + 1 = 0$

by general formula

$e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$

$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$

$e^y = \frac{2x \pm 2\sqrt{x^2 - 1}}{2}$

$e^y = x \pm \sqrt{x^2 - 1}$

apply ln *both sides*

$\ln e^y = \ln\left(x \pm \sqrt{x^2 - 1}\right)$

$y = \ln\left(x \pm \sqrt{x^2 - 1}\right)$

$\cosh^{-1} x = \ln\left(x \pm \sqrt{x^2 - 1}\right)$

This is the expression for $\cosh^{-1} x$ as just a relation and not a function.
For $\cosh^{-1} x$ being in function
$\cosh^{-1} x = \ln\left(x \pm \sqrt{x^2 - 1}\right)$
from
$\cosh^{-1} x = \ln\left(x \pm \sqrt{x^2 - 1}\right)$

$\therefore \cosh^{-1}x = \ln\left(x+\sqrt{x^2-1}\right)$ or $\cosh^{-1}x = \ln\left(x-\sqrt{x^2-1}\right)$

$= \ln\left(x-\sqrt{x^2-1}\right)\left(\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right)$

$= \ln\left(\frac{x^2-x^2+1}{x+\sqrt{x^2-1}}\right)$

$\ln\left(\frac{1}{x+\sqrt{x^2-1}}\right)$

$= \ln\left(x+\sqrt{x^2-1}\right)^{-1}$

$= -\ln\left(x+\sqrt{x^2-1}\right)$

$\therefore \cosh^{-1}x = \pm\ln\left(x+\sqrt{x^2-1}\right)$

$similarly$

$\cosh^{-1}x = \pm\ln\left(x-\sqrt{x^2-1}\right)$

(c c) $y = \tanh^{-1}x$

$\tanh y = x$

$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x$

Multiply by e^y *both sides*

$\frac{e^{2y}-1}{e^{2y}+1} = x$

$e^{2y}-1 = x(e^{2y}+1)$

$e^{2y}-1 = xe^{2y}+x$

$e^{2y}-xe^{2y} = x+1$

$e^{2y}(1-x) = 1+x$

$e^{2y} = \frac{1+x}{1-x}$

$\ln e^{2y} = \ln\left(\frac{1+x}{1-x}\right)$

$2y = \ln\left(\frac{1+x}{1-x}\right)$

$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\therefore y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\therefore \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Examples

1. (i) $\tanh^{-1} x + \tanh^{-1} y = \tanh^{-1}\left(\frac{x+y}{1+xy}\right)$

(ii) $\tanh^{-1} x - \tanh^{-1} y = \tanh^{-1}\left(\frac{x-y}{1-xy}\right)$

Solution (i)

$let\ \tanh^{-1} x = A$

$x = \tanh A$

$let\ \tanh^{-1} y = B$

$y = \tanh B$

$$\tanh(A+B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$$

$$\tanh(A+B) = \frac{x+y}{1+xy}$$

$$A + B = \tanh^{-1}\left(\frac{x+y}{1+xy}\right)$$

$$\therefore \tanh^{-1} x + \tanh^{-1} y = \tanh^{-1}\left(\frac{x+y}{1+xy}\right)$$

Solution (ii)

$let\ \tanh^{-1} x = A$

$x = \tanh A$

$let\ \tanh^{-1} y = B$

$y = \tanh B$

$\therefore \tanh^{-1} x - \tanh^{-1} y = A - B$

$$\tanh(A-B) = \frac{\tanh A - \tanh B}{1 - \tanh A \tanh B}$$

$$\tanh(A-B) = \frac{x-y}{1-xy}$$

$$A - B = \tanh^{-1}\left(\frac{x-y}{1-xy}\right)$$

$$\therefore \tanh^{-1} x - \tanh^{-1} y = \tanh^{-1}\left(\frac{x-y}{1-xy}\right)$$

2. Prove that $\tanh^{-1}\left(\frac{x^2-a^2}{x^2+a^2}\right) = \ln\left(\frac{x}{a}\right)$

Solution

$$\tanh^{-1}\left(\frac{x^2-a^2}{x^2+a^2}\right)=P$$

$$\tanh P=\frac{x^2-a^2}{x^2+a^2}$$

$$\frac{e^P-e^{-P}}{e^P+e^{-P}}=\frac{x^2-a^2}{x^2+a^2}$$

$$\frac{\frac{e^P}{e^{-P}}-\frac{e^{-P}}{e^{-P}}}{\frac{e^P}{e^{-P}}+\frac{e^{-P}}{e^{-P}}}=\frac{\left(\frac{x}{a}\right)^2-\frac{a^2}{a^2}}{\frac{x^2}{a^2}+\frac{a^2}{a^2}}$$

$$\frac{e^{2P}-1}{e^{2P}+1}=\frac{\left(\frac{x}{a}\right)^2-1}{\left(\frac{x}{a}\right)^2+1}$$

by composition

$$e^{2P}=\left(\frac{x}{a}\right)^2$$

$$\ln(e^{2P})=ln\left(\frac{x}{a}\right)^2$$

$$2P=2\ln\left(\frac{x}{a}\right)$$

$$P=\ln\left(\frac{x}{a}\right)$$

$$\therefore \tanh^{-1}\left(\frac{x^2-a^2}{x^2+a^2}\right)=\ln\left(\frac{x}{a}\right) \quad proved$$

3. If $\tanh^{-1}u+\tanh^{-1}v=\frac{1}{2}\ln 5\ \ prove\ that\ V=\frac{2-3u}{3-2u}$

Solution

$$let\ \tanh^{-1}u=A$$

$$\tanh A=u$$

$$\tanh^{-1}v=B$$

$$\tanh B=V$$

$$\therefore A+B=\frac{1}{2}\ln 5$$

$$\tanh(A+B)=\tanh\left(\ln\sqrt{5}\right)$$

$$\frac{\tanh A+\tanh B}{1+\tanh A\tanh B}=\frac{e^{\ln\sqrt{5}}-e^{-\ln\sqrt{5}}}{e^{\ln\sqrt{5}}+e^{-\ln\sqrt{5}}}$$

$$\frac{u+v}{1+uv}=\frac{e^{\ln\sqrt{5}}-e^{\ln\frac{1}{\sqrt{5}}}}{e^{\ln\sqrt{5}}+e^{-\ln\sqrt{5}}}$$

$$\frac{u+v}{1+uv}=\frac{e\left(\ln\sqrt{5}\right)^2-1}{e\left(\ln\sqrt{5}\right)^2+1}$$

$$\frac{u+v}{1+uv}=\frac{5-1}{5+1}$$

$$\frac{u+v}{1+uv}=\frac{4}{6}$$

$$\frac{u+v}{1+uv}=\frac{2}{3}$$

$$3(u+v)=2(1+uv)$$

$$3u+3v=2+2uv$$

$$3v-2uv=2-3u$$

$$v=\frac{2-3u}{2-2u}\quad \textit{hence shown}$$

4.. Given that

$a\cosh t+b\tanh t=R\cosh\,(t+\alpha)$ *where* $a>b>0$ *show that* $\alpha=\frac{1}{2}\ln\left(\frac{a+b}{a-b}\right)$

<u>Solution</u>

$$a\cosh t+b\sinh t=[R\cosh(t+\propto)]$$

$$=R[\cosh t\cosh\alpha+\sinh t\sinh\alpha]$$

$$a\cosh t+b\sinh t=R\cosh t\cosh\propto+R\sinh t\sinh\propto$$

by comparison

$$\frac{b}{a}=\frac{R\sinh\alpha}{R\cosh\alpha}$$

$$\frac{b}{a}=\tanh\alpha$$

$$\alpha = \tanh^{-1}\left(\frac{b}{a}\right)$$

$$from\ \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\alpha = \frac{1}{2}\left(\frac{1+\frac{b}{a}}{1-\frac{b}{a}}\right)$$

$$\alpha = \frac{1}{2}\ln\left(\frac{a+b}{a-b}\right) \quad hence\ shown$$

R- FORMULAE

$$\operatorname{asinh} x + bcosh\ x = \frac{R}{\sqrt{a^2 - b^2}}\operatorname{asinh} x + R)$$

$$\operatorname{asinh} x + bcosh\ \propto = R(sinhxcosh \propto + sinh \propto cosh\ x)$$

$$\operatorname{asinh} x + bcosh\ \propto = Rsinhx\ cosh \propto + Rsinh \propto cosh\ x)$$

By comparison

$$\frac{b}{a} = \frac{R\sinh\propto}{R\cosh\propto}$$

$$\frac{b}{a} = \tanh\propto$$

$$\alpha = \tanh^{-1}\frac{b}{\propto}$$

$$also, coth^2\propto - 1 = cosech^2\propto$$

$$\left(\frac{a}{b}\right)^2 - 1 = cosech^2 \propto$$

$$\frac{a^2 - b^2}{b^2} = cosech^2 \propto$$

$$\frac{b^2}{a^2 - b^2} = sinh^2 \propto$$

$$\sqrt{\frac{b^2}{a^2 - b^2}} = \sinh \propto$$

$$\therefore \sinh \propto = \frac{b}{\sqrt{a^2 - b^2}} \dots \dots . . (i)$$

$$similarly \quad 1 - tanh^2 \propto = sech^2 \propto$$

$$1 - \left(\frac{b}{a}\right)^2 = sech^2 \propto$$

$$\frac{a^2 - b^2}{a^2} = sech^2 \propto$$

$$\frac{a^2}{a^2-b^2} = cosh^2 \propto$$

$$\frac{a}{\sqrt{a^2-b^2}} = \cosh \propto \dots \dots .(ii)$$

$substitute\ (i) and\ (ii) in\ equation\ 1$

$$\text{asinh}\, x + bcosh\, x = \frac{R}{\sqrt{a^2-b^2}}\text{asinh}\, x + \frac{R}{\sqrt{a^2-b^2}} bcosh\, x$$

$$a \sinh x + b \cosh x = \frac{R}{\sqrt{a^2-b^2}}(a \sinh x + b\cosh x)$$

$divide\ by\ a \sinh x + b \cosh x$

$$\frac{R}{\sqrt{a^2-b^2}} = 1$$

$$R = \sqrt{a^2-b^2}$$

$$\therefore a \sinh x + b \cosh x = R \sinh(x + \propto)$$

$$a \sinh x - b \cosh x = R \sinh(x - \propto)$$

$$where \propto = \tanh^{-1}\frac{b}{a}\ \ and\ R = \sqrt{a^2-b^2}$$

$in\ similar\ approach$

$$a \cosh \alpha + b \sinh x = R \cosh(x + \propto)$$

$$a \cosh x - b \sinh x = R \cosh(x - \propto)$$

$$where\ \ = \tanh^{-1}\frac{b}{a}\ \ and\ R = \sqrt{a^2-b^2}$$

Examples

Find the maximum value of

3 cosh$x$ + 2sinh∝

<u>Solution</u>

$3\cosh x + 2\sinh x = \sqrt{3^2 - 2^2}\cosh(x+\alpha)$

$\sqrt{9-4}\cosh(x+\alpha)$

$\sqrt{5}\cosh(x+\alpha)$

$but\ for\ minimum\ \cosh(x+\alpha) = 1$

$\therefore\ minimum\ value\ is\ \sqrt{5}.1 = \sqrt{5}$

$\therefore\ the\ minimum\ value\ of\ \ 3\cosh x + 2\sinh x = \sqrt{5}$

<u>CALCULUS OF HYPERBOLIC FUNCTION</u>

$if\ f(x) = \cosh x$

$f(x) = \dfrac{e^x + e^{-x}}{2}$

$f(x) = \dfrac{e^x}{2} + \dfrac{e^{-x}}{2}$

$f'(x) = \dfrac{e^x}{2} + \ -\dfrac{e^{-x}}{2}$

$= \dfrac{e^x}{2} - \dfrac{e^{-x}}{2}$

$f'(x) = \dfrac{e^x - e^{-x}}{2}$

$but\dfrac{e^x - e^{-x}}{2} = \sinh x$

$\dfrac{d(\cosh x)}{dx} = \sinh x$

$similarly$

$$if\ f(x) = \frac{e^x - e^{-x}}{2}$$
$$f'(x) = \frac{e^x}{2} - -\frac{e^{-x}}{2}$$
$$f'(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$$
$$f'(x) = \frac{e^x + e^{-x}}{2}$$
$$but \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d(\sinh x)}{dx} = \cosh x$$
$$from\ the\ above\ results$$
$$\int \sinh x\, dx = \cosh x + k$$
$$\int \cosh x\, dx = \sinh x + k$$

All other types of hyperbolic functions are differentiated or integrated by the concept of the above results

Note;

In calculus of the hyperbolic functions of the Osborn's rule never operated.

<u>Examples</u>

→Differentiate with respect to x

a) $cosech\ x$

b) $tanh^2 x$

<u>Solution (a)</u>

$$cosech\ x = \frac{1}{\sinh x}$$
$$= (\sinh x)^{-1}$$
$$\therefore \frac{d(cosech\ x)}{dx} = \frac{d(\sinh x)^{-1}}{dx}$$

$= -1(\sinh x)^{-2}\frac{d}{dx}\sinh x$
$= -1(\sinh x)^{-2}\cosh x$

$= \frac{-\cosh x}{sinh^2 x}$

$= \frac{-\cos hx}{\sin h\, x \sin hx}$

$\therefore \frac{d(cosech\ x)}{dx} = -\coth x\ cosech\ x$

→Differentiate

a) $y = \sinh^{-1} x$

b) $y = \cosh^{-1} x$

Solution (b)

$y = \sinh^{-1} x$
$\sinh y = x$
$\cosh y \frac{dy}{dx} = 1$
$\frac{dy}{dx} = \frac{1}{\cosh y}$
$\frac{dy}{dx} = \frac{1}{\sqrt{1 + sinh^2 y}}$
$= \frac{1}{\sqrt{x^2 + 1}}$
$\therefore \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}$
$Similarly$

$(b)\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$

→Evaluate $\int \tanh x\, dx$
solution

$$\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x} dx$$

$$by \int \frac{f'(x)}{f(x)} dx$$

$$\therefore \int \tanh x = \ln(\cosh x) + k$$

→ Evaluate $\int \sinh 5x\, dx$

Solution

$$let\ u = 5x$$

$$du = 5dx$$

$$\frac{du}{5} = dx$$

$$\int \sinh 5x\, dx = \int \sinh u\, .\frac{du}{5}$$

$$= \frac{1}{5}\int \sinh u\, du$$

$$= \frac{1}{5}\cosh u + k$$

$$\therefore \int \sinh 5x\, dx = \frac{1}{5}\cosh u + k$$

→Evaluate $\int \cosh 3x \cosh 2x\, dx$

Solution

$\int \text{Cosh}\, 3x \cosh 2x\, dx$

$$\int \cosh 3x \cosh 2x\, dx = \int \frac{2}{2}\cosh\frac{3x+2x}{2}\cos\frac{3x-2x}{2}dx$$

$$= \frac{1}{2}\int (\cosh 5x + \cosh x)dx$$

$$= \frac{1}{2}\int \cosh 5x\, dx + \frac{1}{2}\int \cosh x\, dx$$

$$= \frac{1}{2}.\frac{\sinh 5x}{5} + \frac{1}{2}.\sinh x + k$$

$$= \frac{1}{10}\sinh 5x + \frac{1}{2}\sinh x + k$$

$$\therefore \int \text{Cosh}\, 3x \cosh 2x \, dx = \frac{1}{10}\sinh 5x + \frac{1}{2}\sinh x + k$$

$\rightarrow \int cosh^2 x dx$

Solution

$$\int cosh^2 x dx = \int \frac{\cosh 2x + 1}{2} dx$$

$$= \frac{1}{2}\int (\cosh 2x + 1) dx$$

$$= \frac{1}{2}\int \cosh 2x + \frac{1}{2}\int 1\, dx$$

$$= \frac{1}{4}\sinh 2x + \frac{1}{2}x + k$$

QUESTIONS

1) Express $\cosh 2x$ and $\sinh 2x$ into exponential form and hence solve $2\cosh 2x - \sinh 2x = 2$
2) Given that $\sinh x = \tan y$. Show that $x = \ln(\tan y \pm \sec y)$
3) Prove that $\tanh^{-1}\left[\frac{x^2-1}{x^2+1}\right] = \ln x$
4) Solve for real values of x. $3\cosh 2x = 3 + \sinh 2x$
5) Prove that $\frac{1+\tanh x}{1-\tanh x} = e^{2x}$

6) (a) If $t = \tanh\frac{x}{2}$, prove that $\sinh x = \frac{2t}{1-t^2}$
(b) use the result in (a) to solve the equation $7\sinh x + 20\cosh x = 24$

7) If $x = \ln\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ find e^x and e^{-x} and hence show that $\sinh x = \tan\theta$
8) Prove that $3\sinh\theta + 4sinh^3\theta$
9) If $\tan\frac{x}{2} = tanAtanB$. Prove the fact that $\tan x = \frac{\sin 2A \sinh 2B}{1+\cos 2A \cosh 2B}$

10) Find the coordinates of the point of intersection of the graph $y = \text{arcsinh}\, x$ and $y = \text{arccosh}(x + 2)$

11) If $\tanh 2y = \frac{-4}{5}$ show that $y = -\frac{1}{2}\ln 3$ and find the value of $\tanh y$

12) Show that the curve $y = \cosh 2x - 4\sinh x$ has just one stationary point and find its coordinates and determine its nature.

13) Prove that $16sinh^2 x cosh^3 x = \cosh 5x + \cosh 3x - 2\cosh x$

14) Prove the fact that $\text{arcsinh}\frac{3}{4} + \text{arcsinh}\frac{5}{12} = \text{arcsinh}\frac{4}{3}$

15) Express $arccosech\ x$ in logarithmic form hence solve the equation $arccosech\ x + \ln x = \ln 3$

16) Show that $\text{arcsinh}\, x = \text{arcsech}\, x$ has only one root and its root is $\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{2}}$

17) Show that $\frac{1+tanh^2 x}{1-tanh^2 x} = \cosh 2x$

STATISTICS 1

Is the branch of Mathematics which deals with the collection, presentation and analysis of data obtained from various experiments.

FREQUENCY DISTRIBUTION

Is the arranged data summarized by distributing it into classes or categories with their frequency

Example;

Variables	20	40	50	60	70
Frequency	2	7	5	4	3

GRAPHICAL REPRESENTATION

Is often useful to represent frequency distribution by means of diagrams.There are different types of diagrams. These are;

1. Line graph

2. Cumulative frequency curve (Ogive)

3. Circles or Pie

4. Bar chart (Histogram)

5. Frequency polygon

TYPES OF DATA

They are;

i) Ungrouped.

ii) Grouped data.

UNGROUPED DATA

Is the type of data in which each value is taken separately which represent to each other e.g. 20, 30 ,40 ,50 etc

REPRESENTATION OF UNGROUPED DATA

-Ungrouped data can be represented by;

(a) Frequency distribution table

(b) Cumulative frequency distribution table

(c) Frequency histogram

(d) Frequency polygon

(e) Frequency curve

(f) Cumulative frequency distribution curve (O gives)

A.FREQUENCY DISTRIBUTION TABLE

Is the table of values with their corresponding frequencies

For instance;

Values	20	30	40	50	60	70
frequency	2	3	5	6	2	1

B. CUMULATIVE FREQUENCY TABLE

Is the table of values with their corresponding cumulative frequencies

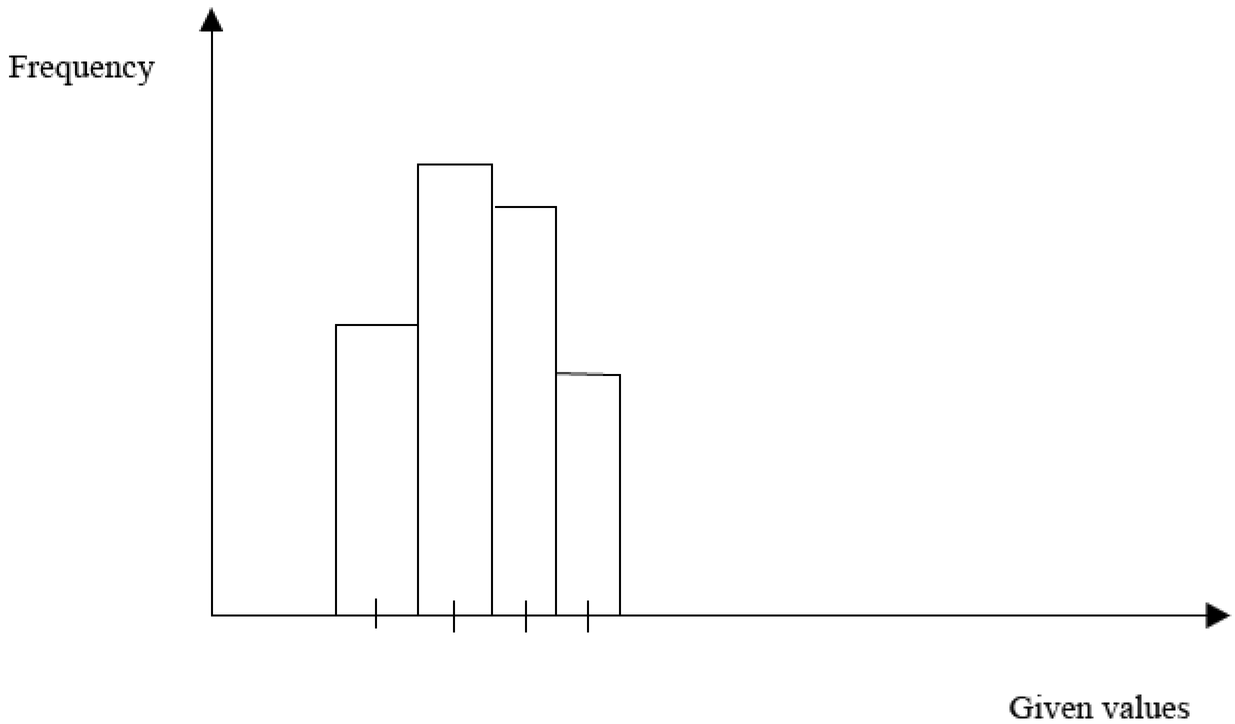
For instance;

Values	20	30	40	50	60	70
frequency	2	5	10	16	18	19

C. FREQUENCY HISTOGRAM

Is the graph which is drawn by using frequency against given values.

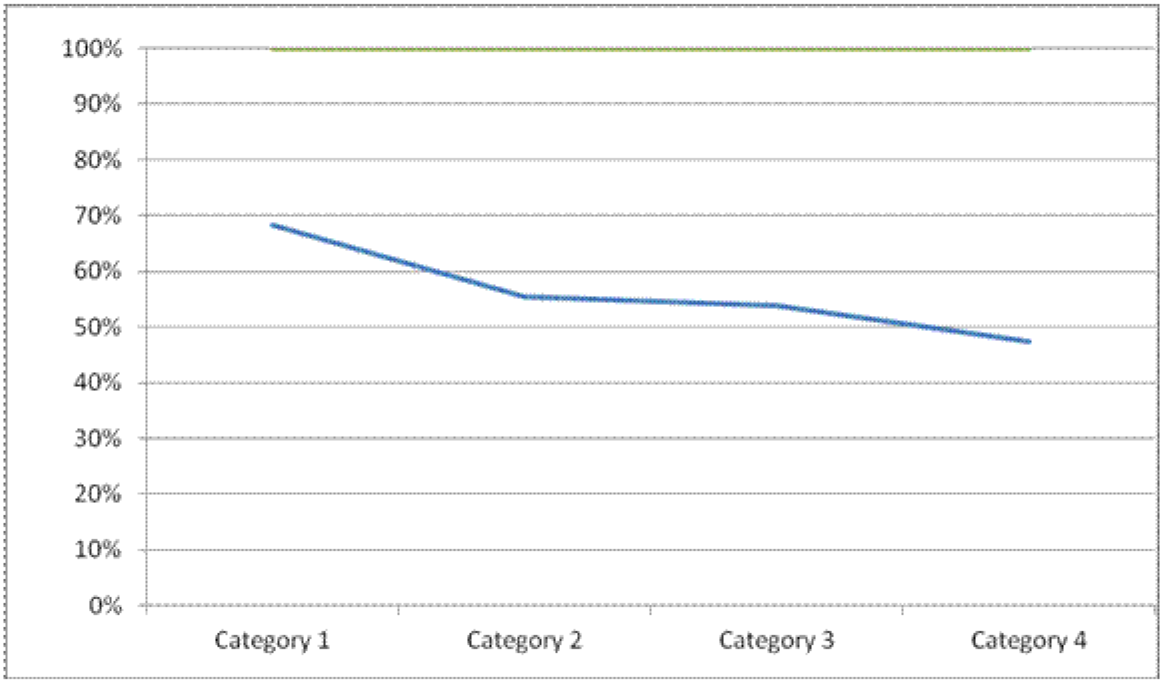
Example



D.FREQUENCY POLYGON

Is the polygon which is drawn by using the corresponding points of frequencies against a given value.

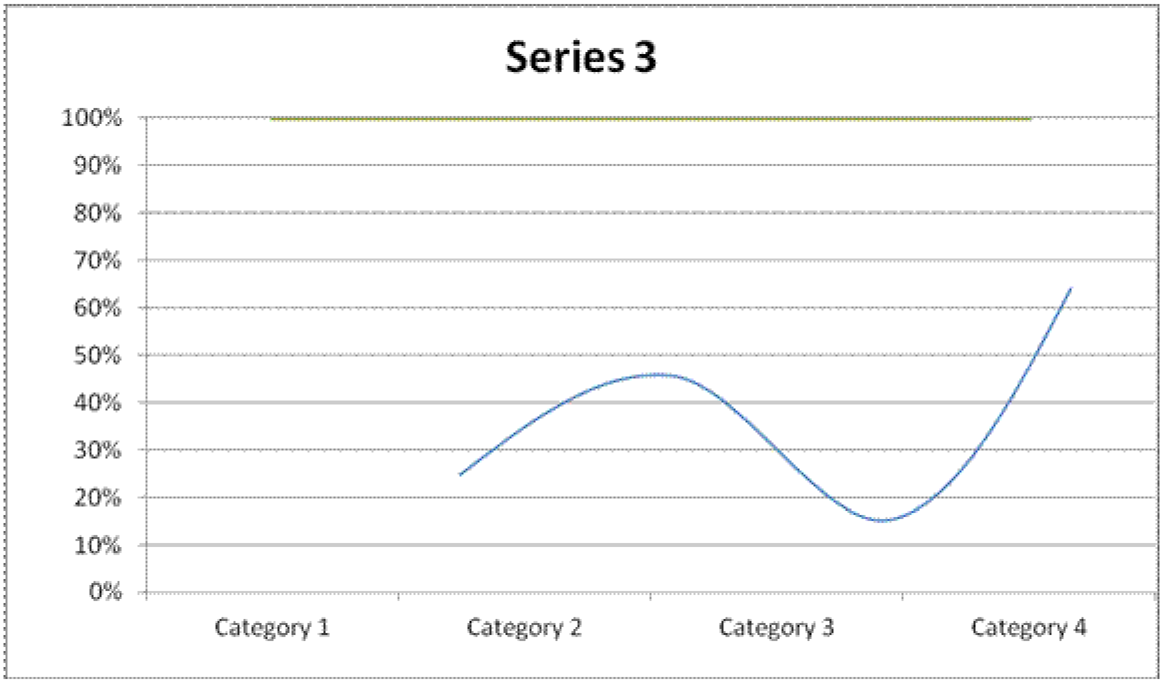
Example



E. THE FREQUENCY CURVE

Is the curve which is drawn by joining (free hand) the corresponding points of frequency against given values.

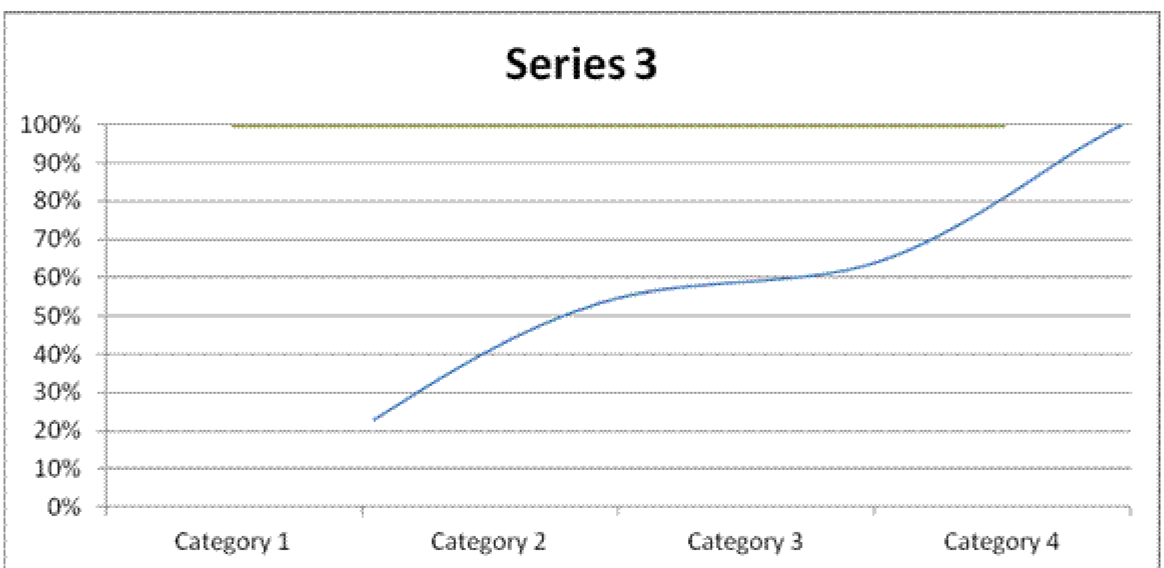
Example



F.CUMULATIVE FREQUENCY CURVE (O GIVE)

Is the curve of cumulative frequency against given values

Example



MEASURES OF CENTRAL TENDENCY

These are;

I) Mean

ii) Mode

iii) Median

iv) Harmonic mean

v) Geometric mean

1**. MEAN** (x)

Is obtained by adding together all the items and dividing by the number of items.

Mean $(\overline{x})$ = $\frac{\sum f(x)}{\sum f}$

2. MODE

Is the number (value) which occur most frequently

For instance

i) Given 2,3,5,5,6,7,7,7,9

→7 has a frequency of 3

→Hence 7 is the mode

ii) Given 1,2,3,4,5,5,6,7,

→then 4 and 5 are the mode their frequency is 2

iii) Given 2, 33, 4, 5, 6, 7= there is no mode at all

3. MEDIAN

Is the middle number (value) when the data is arranged in the order of site.

N.B

I) When the total number of items is ODD say "N" the value of $\frac{1}{2}\ (N+1)^{th}$

Items give the mode.

II) When the total number of items is EVEN, say "N" there are two middle items, then the mean of the values of ½ N^{th} and (½ N+1)th item is the median

4. HARMONIC MEAN

Is the reciprocal of arithmetic mean of their values.

- If it is harmonic mean

Then $\frac{1}{H} = \frac{1}{N}\left(\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \cdots + \frac{1}{X_n}\right)$

$$H = \frac{1}{\frac{1}{N}\left(\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \cdots + \frac{1}{X_n}\right)}$$

$$H = \frac{N}{\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \cdots + \frac{1}{X_n}}$$

For value of x_1, x_2 ,x_3…x_n with the frequency, f1,f2, f_3+…….. f_n respectively

The harmonic mean is given by

$$H = \frac{N}{\sum f\left(\frac{1}{x}\right)}$$

5. GEOMETRIC MEAN

For the values of x_1, x_2, x_3, x_4,………x_n then the geometric mean is given by

$$G = (X_1 \times X_2 + X_3 \ldots X_N)^{\frac{1}{N}}$$

$$G = \sqrt[N]{X_1 \times X_2 + X_3 \ldots X_N}$$

Examples

Find the mean of 20, 22, 25, 28, and 30

Solution

Given 20, 22, 25, 28, 30

$\sum f = 5$

Mean ($\overline{x}$) = $\frac{\sum f(x)}{\sum f}$

$$(\overline{x}) = \frac{20 + 22 + 25 + 28 + 30}{5}$$

$(\overline{x})$ =25

2. Find the mean of the following;

No	8	10	15	20
f	5	8	8	4

Solution

Consider the table below;

No	F	Fx
8	5	40
10	8	80
15	8	120
20	4	80

$\sum f=25$ $\sum f(x)= 320$

Mean $(\overline{x}) = \frac{\sum f(x)}{\sum f}$

$$\overline{x} = \frac{320}{25} = 12.8$$

3. Find the median of 6, 8, 9, 10, 11, 12 and 13.

Solution

Given 6, 8, 9, 10, 11, 12, 13

N=7

Position = $\frac{1}{2}(N+1)^{th}$

= $\frac{1}{2}(7+1)^{th}$

= 4^{th}

Median is 10

4. Find the mode of the following items

0,1,6,7,2,3, 7,6,6,2,6,0, 5, 6, 0

Solution

From the given data 6 has a frequency of 5

→6 is the mode

5. Find the geometric mean of 4,16,8

Solution

Number 4,8,16

N=3

From

$G = \sqrt[N]{X_1 \times X_2 + X_3 \ldots X_N}$

$$G = \sqrt[3]{4 \times 8 \times 16}$$

G= 8

→Geometric mean is 8

6. Calculate the harmonic mean of the data 4,8,16

Solution

Numbers4, 8, 16

N=3

$$H = \frac{N}{\sum f\left(\frac{1}{x}\right)}$$

$$H = \frac{3}{\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}$$

H = 6.857

The harmonic mean =6.857

B.MEASURES OF DISPERSION (Variability)

These are;

i) Variance

ii) Standard deviation

iii) Mean deviation

1. **VARIANCE**

This is given by;

$$var(x) = \frac{\sum f(x-\overline{x})^2}{\sum f}$$

The formula is used for ungrouped data

Also,

$$\text{Var(x)} = \frac{\sum f(x-\overline{x})^2}{\sum f}$$

$$= \frac{\sum f(x^2 - x\overline{x} - x\overline{x} + \overline{x^2})}{\sum f}$$

$$= \frac{\sum fx^2}{\sum f} - \frac{2\overline{x}\sum f\, x}{\sum f} - \frac{\overline{x}^2 \sum f}{\sum f}$$

$$= \frac{\sum fx^2}{\sum f} - 2\overline{x}\overline{x} + \overline{x}^2$$

$$= \frac{\sum fx^2}{\sum f} - 2\overline{x}^2 + \overline{x}^2$$

$$= \frac{\sum fx^2}{\sum f} - \overline{x}^2$$

$$= \frac{\sum fx^2}{\sum f} - \left(\frac{\sum f\,(x)}{\sum f}\right)^2$$

$$var(x) = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum f\,(x)}{\sum f}\right)^2$$

2. STANDARD DEVIATION

This is given by;

S.D = $\sqrt{var(x)}$

S.D= $\sqrt{\frac{\Sigma f(x-\overline{x})^2}{\Sigma f}}$

S.D $= \sqrt{\frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f(x)}{\Sigma f}\right)^2}$

3. MEAN DEVIATION

This is given by

M.D = $\frac{\Sigma f/x-\overline{x}/}{\Sigma f}$

Also f =1

M.D = $\frac{\Sigma f/x-\overline{x}/}{\Sigma f}$

Examples

7. From the distribution 1, 2, 3, 4, 5 find variance

8. Given the distribution 2, 3, 4, 5, 6, 7, 4, 5, 3 find

a) Variance

b) Standard deviation

c) Mean deviation

Solution

Consider the distribution table

x	f	$(x-\bar{x})$	$\lvert x-\bar{x}\rvert$	$(x-\bar{x})^2$	$\sum f(x-\bar{x})$	$\sum f(x-\bar{x})^2$
2	1	-2.3	2.3	5.29	2.3	5.29
3	2	-1.3	1.3	1.69	2.6	3.38
4	2	-0.3	0.3	0.09	0.6	0.18
5	2	0.7	0.7	0.49	1.4	0.98
6	1	1.7	1.7	2.89	1.7	2.89
7	1	2.7	2.7	7.29	2.7	7.29
	$\sum f = 9$	$\sum(x-\bar{x}) = 1.2$	$\sum\lvert x-\bar{x}\rvert = 9$	$\sum(x-\bar{x})^2 = 17.74$	$\sum f(x-\bar{x}) = 11.3$	$\sum f(x-\bar{x})^2 = 20.01$

$$\bar{x} = \frac{2\times 1 + 3\times 2 + 4\times 2 + 5\times 2 + 6\times 1 + 7\times 1}{9}$$

$$\bar{x} = 4.3$$

Variance

Form

$$variance = \frac{\Sigma f(x-\bar{x})^2}{\Sigma f}$$

$$= \frac{20.01}{9}$$

Var(x) = 2.223.

b) Standard deviation

S.D = $\sqrt{variance}$

S.D = $\sqrt{2.223}$

S.D = 1.491

c). Mean deviation

$= \frac{\Sigma f/x-\bar{x}/}{\Sigma f}$

$= \frac{11.3}{9}$

= 1.25556

C.MEASURES OF POSITION

These are;

i) Quartile

ii) Decile

iii) Percentile

1. QUARTILE

This is the division of frequency distribution into four equal parts.

Hence there are;

1st –quartile

2nd-quartile

3^{rd-}quartile

-The position of 1st quartile is $\frac{n}{4}$

-The position of 2nd quartile is $\frac{2n}{4}$

-The position of 3rd quartile is $\frac{3n}{4}$

N.B. 2nd quartile is the median

INTER-QUARTILE RANGE

This is the different between 1st and 3rd quartile

Mathematically;

Inter-quartile =3rd quartile – 1st quartile

I.Q.R = $Q_3 - Q_1$

SEMI- INTER QUARTILE RANGE

This is given as a half of inter-quartile range

i.e. semi inter quartile range

Semi I.Q.R= $Q_3 - Q_1$

2. DECILE

This is the division of frequency distribution into ten equal parts

Hence there are

1st Decile

↓

2nd Decile

↓

3rd Decile

↓

5th Decile

↓

9th Decile

The position of 1st decile is $\frac{n}{10}$

The position of 2nd decile is $\frac{2n}{10} = \frac{n}{5}$

The position of 3rd decile is $\frac{3n}{10}$

The position of 5rd decile is $\frac{5n}{10} = \frac{n}{2}$

The position of 9th decile is $\frac{9n}{10}$

N.B
The 5th decile is the median

PERCENTILE

This is the division of frequency distribution into 100, equal parts, hence these are;

1st percentile

2nd percentile
3rd percentile
4th percentile

50th percentile

99th percentile

The position of 1st percentile is $\frac{n}{100}$

The position of 2nd percentile is $\frac{2n}{100} = \frac{n}{50}$

The position of 3rd percentile is $\frac{3n}{100}$

The position of 50th percentile is $\frac{50n}{100} = \frac{n}{2}$

The position of 99th percentile is $\frac{99n}{100}$

N.B. 50th Percentile is the median

Examples

Given the distribution 2, 3, 4, 5, 6, 7, 8, 9 find

i) 1st quartile

ii) 2nd quartile

iii) 3rd quartile

iv) Inter- quartile range

v) Semi inter-quartile range

<u>Solution</u>

Given 2, 3, 4, 5, 6, 7, 8, 9

n=8

i) 1st quartile (Q_1)

Position of (Q_1) = $\frac{n}{4}$

= $\frac{8}{4} \rightarrow$ 2nd Value

1st quartile = $\frac{3+4}{2} = 3.5$

ii) second quartile (Q_2)

Position of Q_2 = $\frac{n}{2}$

= $\frac{8}{2} \rightarrow$ 4th value

2nd quartile = $\frac{5+6}{2} = 5.5$

iii)Third quartile (Q_3)

Position of Q_3 = $\frac{3n}{4}$

$$3 \times \frac{8}{4} = 6$$

3rd quartile = $\frac{7+8}{2} = 7.5$

iv) Inter-quartile range

I.Q.R = $Q_3 - Q_1$

7.5 − 3.5 = 4

v) Semi inter-quartile range

S.I.Q$_R$ $= \frac{Q_3 - Q_1}{2} = \frac{4}{2} = 2$

QUESTIONS

3. Given the distribution 1, 2, 3, 4, 5, 6, 7, 8, 9 find

i) Quartile 1

ii) Quartile 2

iii). Quartile 3

4. Given the distribution 2, 3, 5, 6, 7, 8, 9, 9, 10, 11, 12 find.

a) Lower quartile

b) Median

c) Upper quartile

5. From the distribution 20, 23, 23, 26, 27, 28 find

i) Q_1 ii)Q_2 iii)Q_3

6. Given the distribution 147,150,154,158,159,162,164,165 find i) Q_1 ii) Q_2 iii) Q_3

7. Given the frequency distribution 10,12,13,15,17,19,24,26,26, find i)Q_1 ii) Q_2 iii) Q_3

8. From the frequency distribution 2,3,4,5,6,7,8,9,10,11,12

Find i) first decile

ii) 4^{th} decile

iii) 5^{th} decile

9. The following table below shows the scores obtained when a dice thrown 60 times .Find the median score

SCORE	1	2	3	4	5	6
FREQUENCY(F)	12	9	8	13	9	9

10. Find the median and inter quartile range of the frequency distribution

X	5	6	7	8	9	10
F	6	11	15	18	16	5

GROUPED DATA

This is the grouping of frequency distribution obtained from various experiments.0-9, 10-19, 20-29, 30,-39

TERMINOLOGIES

1 .LOWER CLASS INTERVAL

This is the lower interval or lower limit of the class.

Example

$x_1 \rightarrow x_2$, $x_5 \rightarrow x_6$, $x_7 \rightarrow x_8$

x_1, x_5, x_7 are lower interval of the classes

2. UPPER CLASS INTERVAL

This is the upper interval of upper limit of the class.

E.g. $x_1 \rightarrow x_2$, $x_3 \rightarrow x_4$, $x_5 \rightarrow x_6$

x_2 ,x_4 and x_6 are the upper interval of the classes

3. CLASS MARK(X)

Is the average between class intervals.

Mathematically;

$$\text{Class mark (x)} = \frac{L.C.L+U.C.L}{2}$$

Where L.C.L=lower class limit

U.C.L= upper class limit

4. CLASS BOUNDARIES

These are real limits of the classes

a) Upper class boundaries

i.e. $U.C.B = U.C.B + 0.5$

Where;

U.C.B = upper class boundary

U.C.L = Upper class limit

b) Lower class boundary

L.C.B = L.C.L-0.5

Where;

L.C.B Lower class boundary

L.C.L = Lower class limit

5. CLASS SIZE(WIDTH)

Is the difference between lower or upper and upper class limit or class boundaries between two successive classes.

Mathematically;

$$C = U.C.B - L.C.B$$

$$C = (U.C.L + 0.5) - (L.C.L - 0.5)$$

$$C = U.C.L - L.C.L + 0.5 + 0.5$$

$$C = U.C.L - L.C.L + 1$$

REPRESENTATION OF GROUPED DATA

These are;

i) Frequency distribution table

ii) cumulative frequency distribution table

iii) frequency histogram

iv)Cumulative frequency curve (o give)

v) frequency polygon

vi) scatter diagram

1. FREQUENCY DISTRIBUTION TABLE

Is the table of class interval with their corresponding frequency.

class interval	0-9	10-19	20-29	30-39

frequency	2	7	3	4

How to prepare frequency distribution table

A. IF CERTAIN CLASS INTERVAL PROVIDED

In this case, prepare the frequency table by using interval provided

Complete the column of frequency by random inspection.

→The last interval should end up to where the highest value belong to

B. IF CERTAIN CLASS MARKS PROVIDED

1st method

i) Mark the lowest value from the data provided and call it as lower limit (L) of the first class interval

ii)using the formula of class mark calculate the upper limit(u)

Class mark $= \frac{U+L}{2}$

iii) Indicate the first interval as L to U

The proceed under the same interval in order to get frequency distribution table

2nd method

i) Get class size (c) then find the value of $(\frac{C-1}{2})$

ii) By using the value of $(\frac{C-1}{2})$ get the intervals as follows

Lower interval $= classmark - (\frac{C-1}{2})$

Upper interval $= classmark + (\frac{C-1}{2})$

iii) By using class interval obtained prepare the frequency distribution table

C. IF CLASS MARK AND CLASS INTERVALS ARE NOT PROVIDED

In this case the frequency distribution table should be prepared under the following steps.

i) Perform random inspection of highest and lowest value
ii) Determine the value of the range as follows

Range (R) = $highest(H) - lowest(L)$

R $= H - L$

iii) Determine the class size according to required number of classes by using the formula

$$C = \frac{R}{N}$$

Where r=range

C=class size

N=number of class required

E.g. C= $\frac{93}{10} = 9.3 \approx 10$

C= $\frac{97}{10} = 9.7 \approx 10$

iv) By using class size obtained above, prepare frequency distribution table by regarding condition existing from the data

Examples

1.The following are the results from mathematical test of 20 students 27,21,24,27,31,40,45,46,50,48,38,29,49,98,35,34,44,23,25,49 prepare the following distribution table by using class interval 21-25,26-30,31-35

2. The following are the results of physics test of 50 students at Azania sec school 21,23,48,54,64,77,68,52,31,40,33,43,53,61,71.82,75,61,64,34,25,26,31,32,36,48,45,44,55,52,60, 67,67,7,74,78,80,85,90,97,26,27,37,38,34,39,40.41.45.48 prepare the frequency distribution table using class mark 23,28,33

3. 3 .The data below give time in minutes .it takes a computer to drive to work for a period of lasting 50 days

25,40,27,43,23,28,39,33,29,26,34,32,28,30,39,32,25,27,28,28,27,35,28,46,24,24,22,31,28,27,35,28,46,24,24,22,31,28,27,31,23,32,36,22,26,34,30,27,25,42,25,37,30.27,31, 30, 48, 28, 24

Construct a frequency distribution table having six classes for which 20 is the lowest unit of the first class and 49 is the upper limit of the size of class.

Solution 1

Frequency distribution table

CLASS INTERVAL	FREQUENCY
21-25	4
26-30	3
31-35	3
36-40	2
41-45	2
46-50	6

Solution 2

Given:

Class mark(x) = 23, 28, 33

Class size (i) 28-23=5

$$\frac{C-1}{2}=\frac{5-1}{2}=2$$

→1st class interval 23 – 2 = 21

=21

Upper limit 23 + 2 = 25

=25

= 21→25

→2nd class interval

Lower limit = 28 – 2

=26

Upper limit = 28 + 2

=30

= 26→30

→3rd class interval

Lower limit = 33 – 2

= 31

Upper limit = 33 + 2

=35

Others36-40

41-45

46-50

Frequency distribution table

Class interval	Frequency
21-25	3
26-30	3
31-35	5
36-40	6
41-45	5
46-50	4
51-55	4
56-60	2
61-65	4
66-70	4
71-75	4
76-80	3
81-85	2
86-90	1

Solution 3

Range(R) = H – C

= 49 – 20

= 29

Also

N=number of class required

N=6

C=$\frac{R}{n}$

C = $\frac{29}{6} = 4.85$

1st class interval

C= U.C.B – L.C.D

C = (U.C.L+0.5) ‾ (L.C.L-0.5)

C = U.C.L – L.C.L + 1

5 = U.C.L – 20 + 1

5 = U.C.L – 19

U.C.L = 24

→20-24

Other intervals are 25→29, 30→34, 35→39, 40→44, 45→49

Frequency distribution table

Class interval	Frequency

20-24	8
25-29	20
30-34	12
35-39	5
40-44	3
45-49	2

II. CUMULATIVE FREQUENCY DISTRIBUTION TABLE

Is the table of class interval with their corresponding Cumulative frequency

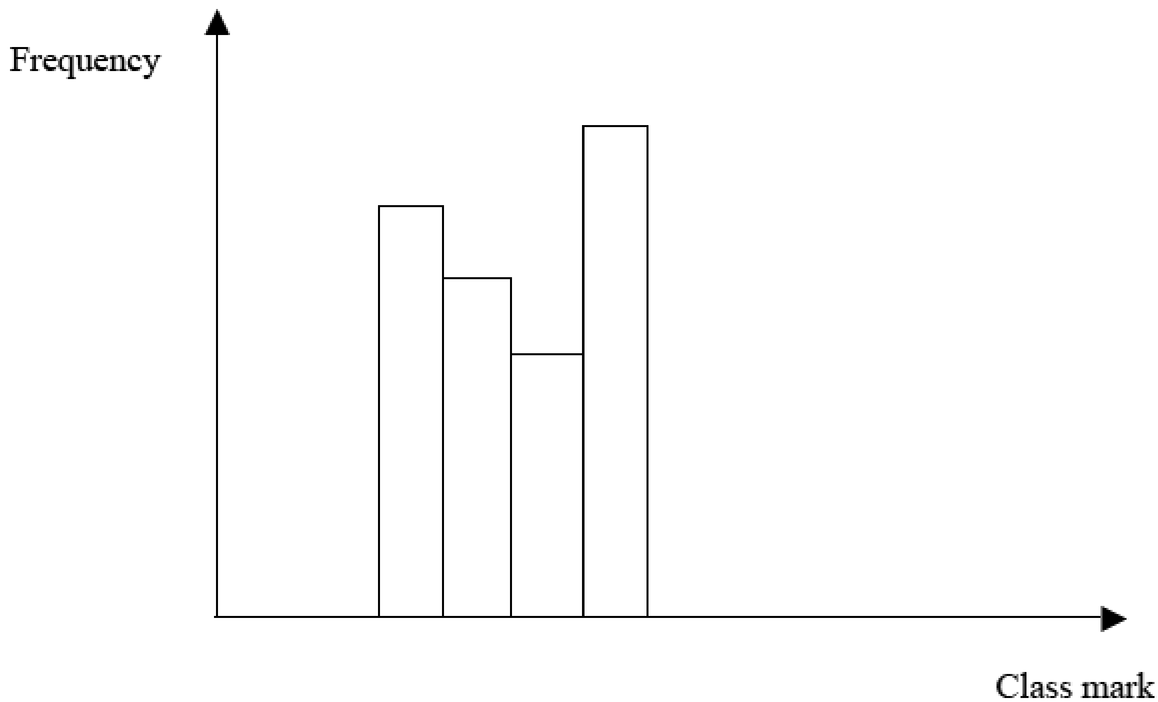
Example

Class interval	0-9	10-19	20-29
Cum freq	3	7	20

III. FREQUENCY HISTOGRAM

Is the graph which is drawn by using frequencies against class mark

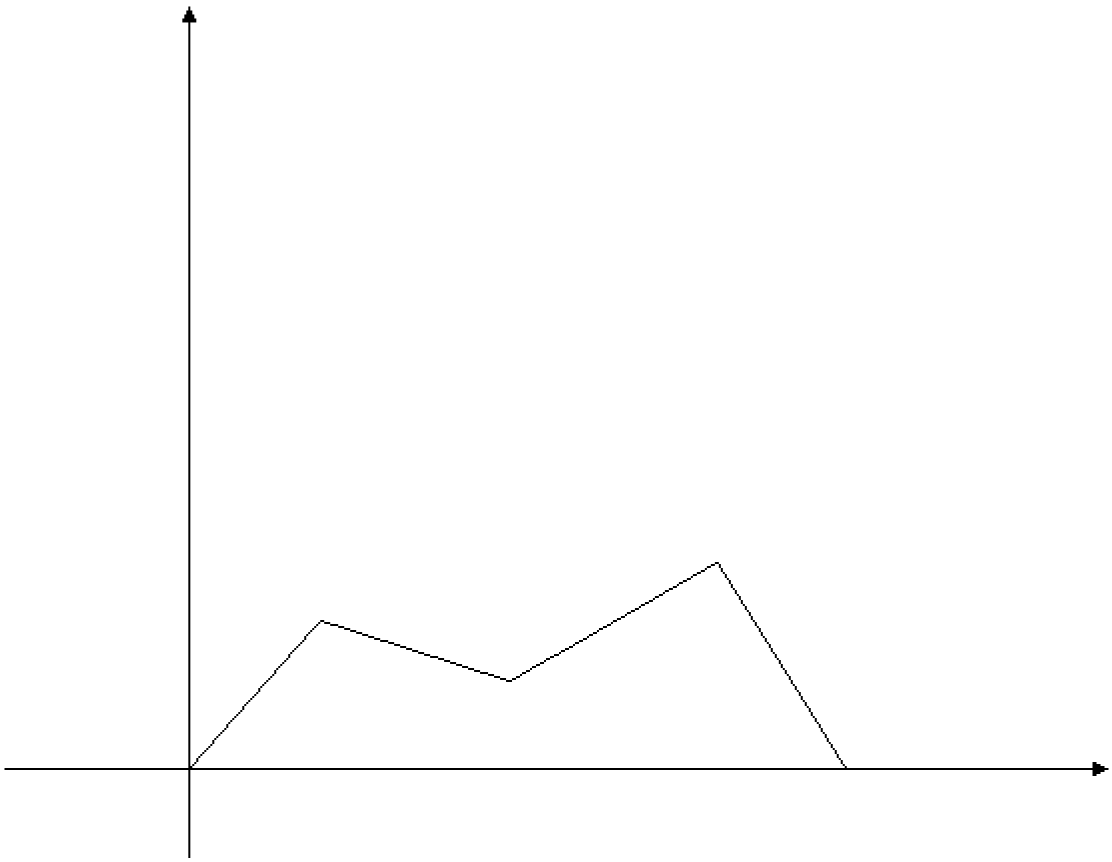
Example



IV) FREQUENCY POLYGON

Is the polygon which is drawn by joining the corresponding points of frequencies against the class marks.

Example;

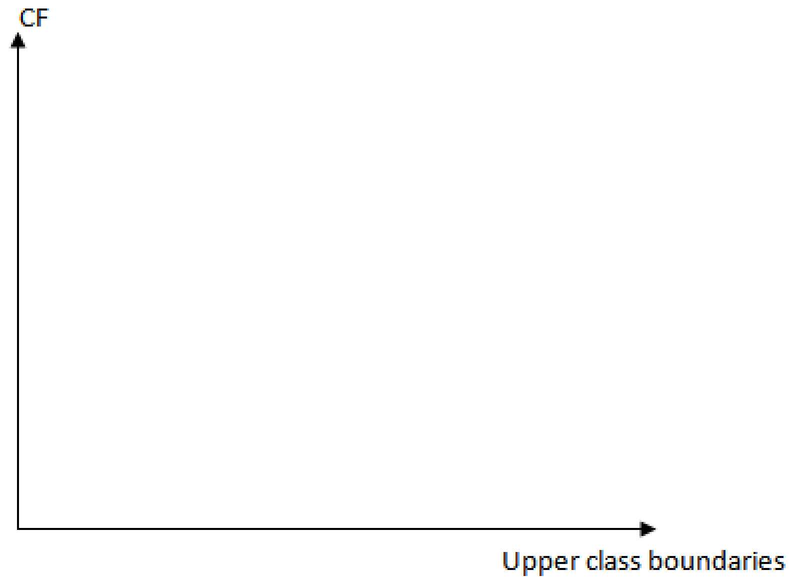


Frequency

V) CUMULATIVE FREQUENCY CURVE (O GIVE)

Is the curve which is drawn by joining (free hand) the corresponding points of cumulative frequencies against the upper class boundary.

Example;



VI) SCATTERED DIAGRAM

Is the diagram where frequencies scattered against class mark without connecting the points.

MEASURES OF GROUPED DATA

These are;

a) Measures of central tendency

b) Measures of dispersion (variability)

c) Measures of position

A.MEASURES OF CENTRAL TENDENCY

These are i) mean

ii) Median

iii) Mode

I. MEAN ($\overline{x}$)

-By direct method

Mean ($\overline{x}$) $= \frac{\sum f(x)}{\sum f}$

Where

∑f(x) → is the sum of frequencies times class mark

∑f →sum of frequency

X	X_1	X_2	X_3	X_n
F	F_1	F_2	F_3	F_n

$$\overline{x} = \frac{x_1 f_1 + x_2 f_2 + x_3 f_3 + \cdots + x_n f_n}{f_1 + f_2 + f_3 + \cdots + f_n}$$

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i}$$

$$\overline{x} = \frac{\sum f(x)}{\sum f}$$

By assumed mean method

-I.e. mean(x) = $\frac{\sum f(x)}{\sum f} \ldots\ldots\ldots.(i)$

$$let\ x - A = d$$

$$x = A + d \ldots\ldots\ldots.(ii)$$

Where

X → is the class mark

A → is the assumed mean

D → deviation

Substitute (ii) into (i)

$$(\mathrm{x}) = \frac{\sum f(x)}{\sum f}$$

$$= \frac{\sum f(A + d)}{\sum f}$$

$$\overline{x} = \frac{A\sum f}{\sum f} + \frac{\sum f(d)}{\sum f}$$

$$\overline{x} = A + \frac{\sum f(d)}{\sum f}$$

BY CODING METHOD

$$mean(\overline{x}) = A + \frac{\sum f(d)}{\sum f} \ldots \ldots \ldots (i)$$

$$put\ u = \frac{d}{c}$$

$$d = cu \ldots \ldots \ldots (ii)$$

Where;

d → is the deviation

C → is the class size

n → is the coding number

Substitute (ii) into (i)

$$\overline{x} = A + \frac{\sum f(d)}{\sum f}$$

$$\overline{x} = A + \frac{\sum f(cu)}{\sum f}$$

$$\overline{x} = A + \frac{c\sum f(u)}{\sum f}$$

$$mean(\overline{x}) = A + \frac{c\sum f(u)}{\sum f} \quad is\ the\ mean\ by\ coding\ method$$

Examples

1. From the following distribution

X	10	20	30	40	50
F	16	18	25	19	22

Find the mean by

i) Direct method

ii) assumed mean

iii) Coding method

Solution

Consider the distribution table

X	F	F(x)	d= x – a	f(d)	u= $\frac{d}{c}$	fu
10	16	160	-20	-320	-2	-32
20	18	360	-10	-180	-1	-18
30	25	750	0	0	0	0
40	19	160	10	190	1	19
50	22	1100	20	440	2	44
	$\sum f = 100$	$\sum f(x) = 2530$		$\sum fd = 130$		$\sum fu = 13$

Let A = Assumed mean

= 30

C =Class size

= 10

a) By direct mean method

I.e. Mean ($\overline{x}$) $= \frac{\sum f(x)}{\sum f}$

$= \frac{2530}{100}$

Mean (x) = 25.30

b) By assumed mean method

$$\overline{x} = A + \frac{\sum f(d)}{\sum f}$$

$$= 30 + \frac{130}{100}$$

$$\overline{x} = 31.30$$

c) By coding method

$$mean(\overline{x}) = A + \frac{c\sum f(u)}{\sum f}$$

$$= 30 + 10\left(\frac{13}{10}\right)$$

$$= 30 + 1.31$$

$$\overline{x} = 31.30$$

MODE

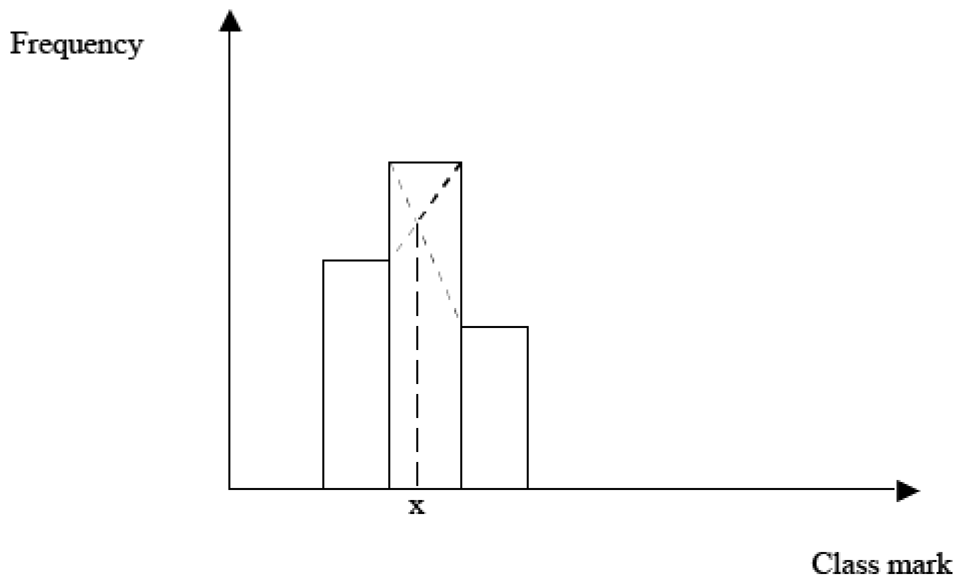
This is the value which occurs most frequently in grouped data mode can be determined by using two methods

a) By estimation from histogram

b) By calculation method

A) MODE FROM HISTOGRAM

-Consider the three bars under consideration of the highest bar with their two adjacent bars from the histogram.



BY CALCULATION METHOD

-Assume that the figure below represents three rectangles of the histogram of the frequency distribution of central rectangle corresponding to modal class.

Where

$D_1 \rightarrow$ is the difference between the frequencies of the mode class and the frequency of the class just before the modal class.

D_2 → is the difference between the frequency of the modal class and the frequency of the class just after the modal class

L_c → is the lower class boundary of the modal class

U → is the upper class boundary of the modal class

Hence from the histogram $\Delta RPQ \sim \Delta SPT$

$$\frac{\overline{EP}}{\overline{PF}} = \frac{\overline{RQ}}{\overline{ST}}$$

$$\frac{\overline{EP}}{\overline{RQ}} = \frac{\overline{PF}}{\overline{ST}}$$

$$\frac{X - L_C}{D_1} = \frac{U - X}{D_2}$$

$D_2(X - L_C) = D_1(U - X)$

$D_2X - D_2L_C = D_1U - D_1X$

$D_1X + D_2X = D_2L_C + D_1U$

$X(D_1 + D_2) = D_2L_C + D_1U$

But

$U - L_C$ = class size(C)

$C = U - L_C$

$U = C + L_C$

$X(D_1 + D_2) = D_2L_C + D_1(L_C + C)$

$$\frac{X_1(D_1+D_2)}{D_1+D_2} = \frac{L_C(D_1+D_2)}{(D_1+D_2)} + \frac{D_1C}{(D_1+D_2)}$$

$$X = L_C + \left(\frac{D_1}{D_1+D_2}\right)C$$

$$MODE = L_C + \left(\frac{D_1}{D_1+D_2}\right)C$$

2. MEDIAN

Position of the median class N/2 hence therefore using interpolation

$$MEDIAN = L_C + \left(\frac{\frac{N}{2} - \sum fb}{fm}\right)C$$

Where;

L_C→ is lower boundary of the median class

$\sum fb$ → Sum of the frequency before the median class

fm→ frequency of the modal class

c → class size

Example

1. from the following distribution table

Class interval	Frequency	Comm. freq
1-7	8	8
8-14	10	18

15-21	22	40
22-28	15	55
29-35	7	62
36-42	18	80

Find i) mode

ii) Median

2. Find the mean and the median o the following distribution

Class interval	Frequency
0.20-0.24	6
0.25-0.29	12
0.30-0.34	19
0.35-0.39	13

3. Given the frequency distribution table below

Class interval	Frequency
16.50-16.59	25
16.60-16.69	47
16.70-16.79	65

16.80-16.89	47
16.90-16.99	16

Solution 1

Consider the distribution table

Class interval	Frequency	Comm. Freq
1-7	8	8
8-14	10	18
15-21	22	40
22-28	15	55
29-35	7	62
36-42	18	80

i) mode from the table

Modal class = 15→21

$$MODE = L_C + \left(\frac{D_1}{D_1 + D_2}\right)C$$

$$but\ \ l_c = 15 - 0.5 = 14.5$$

$$C = (21.05) - (15 - 0.5)$$

$21.05 - 14.5 = 7$

$D_1 = 22 - 10 = 12$

$D_2 = 22 - 15 = 7$

$$MODE = 14.5 + \left(\frac{12}{12+7}\right)7$$

$MODE = 18.92$

ii) median

Position of median class

$= \frac{N}{2} = \frac{80}{2}$

=40

Median class =15→21

L_C= (15 – 0.5)

= 14.5

∑fb=18

Fm=22

C=7

Then

$$MEDIAN = L_C + \left(\frac{\frac{N}{2} - \sum fb}{fm}\right)C$$

$$= 14.5 + \left(\frac{40-18}{22}\right)7$$

Median = 21.5

MEASURES OF DISPERSION (variability)

These are

i) Variance

ii) standard deviation

VARIANCE

→Variance by direct mean method

Recall

$$var(x) = \frac{\sum f\,(x-\overline{x})^2}{\sum f}$$

$$var(x) = \frac{\sum f\,(x-\overline{x})(x-\overline{x})}{\sum f}$$

$$= \frac{\sum f\left(x^2 - 2x\overline{x} + \overline{x}^2\right)}{\sum f}$$

$$= \frac{\sum f\,x^2}{\sum f} - \frac{\sum f 2x\overline{x}}{\sum f} + \frac{\sum f\overline{x}^2}{\sum f}$$

$$= \frac{\sum f\,x^2}{\sum f} - \frac{2\overline{x}\sum f\,x}{\sum f} + \frac{\overline{x}^2\sum f}{\sum f}$$

$$= \frac{\sum f x^2}{\sum f} - 2\overline{x}\overline{x} + \overline{x}^2$$

$$= \frac{\sum f x^2}{\sum f} - 2\overline{x}^2 + \overline{x}^2$$

$$= \frac{\sum f x^2}{\sum f} - (\overline{x})^2$$

$$= \frac{\sum f x^2}{\sum f} - \left(\frac{\sum f x}{\sum f}\right)^2$$

$$var(x) = \frac{\sum f x^2}{\sum f} - \left(\frac{\sum f x}{\sum f}\right)^2 \text{ variance by direct mean method}$$

VARIANCE BY ASSUMED MEAN METHOD

Recall

$$var(x) == \frac{\sum f x^2}{\sum f} - \left(\frac{\sum f x}{\sum f}\right)^2 \dots\dots\dots.(i)$$

Put X – A= d

X = A + d…… (ii)

Where

X → class mark

A → Assumed mean

Substitute……..ii)………..i) as follows

$$var(x) = \frac{\sum f\, x^2}{\sum f} - \left(\frac{\sum f x}{\sum f}\right)^2$$

$$= \frac{\sum f\,(A+d)^2}{\sum f} - \left(\frac{\sum f(A+d)}{\sum f}\right)^2$$

$$= \frac{\sum f\,(A^2 + 2Ad + d^2)}{\sum f} - \left(\frac{\sum f(A+d)}{\sum f}\right)^2$$

$$\frac{\sum f A^2}{\sum f} + \frac{\sum f\, 2Ad}{\sum f} + \frac{\sum f d^2}{\sum f} - \left(\frac{\sum f A}{\sum f} + \frac{\sum f d}{\sum f}\right)^2$$

$$\frac{A^2 \sum f}{\sum f} + \frac{2A\sum f\, d}{\sum f} + \frac{\sum f\, d^2}{\sum f} - \left(\frac{A\sum f}{\sum f} + \frac{\sum f d}{\sum f}\right)^2$$

$$= A^2 + 2A\bar{d} + \bar{d}^2 - \left(A + \bar{d}\right)^2$$

$$= A^2 + 2A\bar{d} + \bar{d}^2 - A^2 - 2A\bar{d} - \bar{d}^2$$

$$\bar{d}^2 - \bar{d}^2$$

$$var(x) = \frac{\sum f d^2}{\sum f} - \left(\frac{\sum f d}{\sum f}\right)^2$$

VARIANCE BY CODING METHOD

Recall,

$$var(x) = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2 \ldots \ldots \ldots (i)$$

$$put\, u = \frac{d}{c} \qquad d = uc \ldots \ldots \ldots (ii)$$

Where;

D = deviation

C = class size

U = coding number

Substitute (ii) into (i)

$$var(x) = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$$

$$var(x) = \frac{\sum f(cu)^2}{\sum f} - \left(\frac{\sum f(cu)}{\sum f}\right)^2$$

$$= \frac{\sum fc^2u^2}{\sum f} - \left(c\frac{\sum fu}{\sum f}\right)^2$$

$$var(x) = c^2\left(\frac{\sum fu^2}{\sum f} - \left(\frac{\sum fu}{\sum f}\right)^2\right) is\ the\ variance\ by\ coding\ method$$

2. STANDARD DEVIATION

This is given by $\sqrt{variance(x)}$

→By direct mean method

$$S.D = \sqrt{\frac{\sum f x^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

→By assumed mean method

$$S.D = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

→By Coding method

$$S.D = C\sqrt{\frac{\sum fu^2}{\sum f} - \left(\frac{\sum fu}{\sum f}\right)^2}$$

questions

1. Given the distribution class interval frequency

Class interval	Frequency
1-5	8
6-10	18
11-15	9
16-20	25

21-25	40

Find the standard deviation by coding method.

2. The table below shows the frequency distribution of intelligence quotient (IQ)of 500 individuals

I.Q	Frequency
82-85	5
86-89	19
90-93	32
94-97	49
98-101	71
102-105	92
106-109	75
110-113	56
114-117	39
118-121	28
122-125	18
126-129	10
130-133	6

Using coding method find

i) Mean

ii) Standard deviation

c. MEASURES OF POSITION

These are

1.quartile

2. deciles

3. Percentile

1.QUARTILE

Recall

$$MEDIAN = L_c + \left(\frac{\frac{N}{2} - \sum fb}{fm} \right) C$$

$$\rightarrow 2^{nd} quartile = L_c + \left(\frac{\frac{N}{2} - \sum fb}{fm} \right) C$$

$$\rightarrow 1^{st}\ quartile = L_c + \left(\frac{\frac{N}{4} - \sum fb}{fQ_1} \right) C$$

$$\rightarrow 3^{rd} quartile = L_c + \left(\frac{\frac{3N}{4} - \sum fb}{fQ_3} \right) C$$

2. DECILE

Recall

$$MEDIAN = L_C + \left(\frac{\frac{N}{2} - \sum fb}{fm}\right)C$$

$$1^{st} decile = L_C + \left(\frac{\frac{N}{10} - \sum fb}{fD_1}\right)C$$

$$2^{nd} decile = L_C + \left(\frac{\frac{2N}{10} - \sum fb}{fD_2}\right)C$$

$$3^{rd} decile = L_C + \left(\frac{\frac{3N}{10} - \sum fb}{fD_3}\right)C$$

$$9^{th} decile = L_C + \left(\frac{\frac{9N}{10} - \sum fb}{fD_9}\right)C$$

3. PERCENTILE

Recall

$$MEDIAN = L_C + \left(\frac{\frac{N}{2} - \sum fb}{fm}\right)C$$

$$50^{th} percentile = L_C + \left(\frac{\frac{N}{2} - \sum fb}{fm} \right) C$$

$$1^{st}\ percentile = L_C + \left(\frac{\frac{N}{100} - \sum fb}{fP_1} \right) C$$

$$2^{nd}\ percentile = L_C + \left(\frac{\frac{2N}{100} - \sum fb}{fP_2} \right) C$$

$$3^{rd}\ percentile = L_C + \left(\frac{\frac{3N}{100} - \sum fb}{fP_3} \right) C$$

$$99^{th}\ percentile = L_C + \left(\frac{\frac{99N}{100} - \sum fb}{fP_{99}} \right) C$$

Note

2^{nd} quartile 5^{th}, decile and 50^{th} percentile are MEDIANS

PROBABILITY DISTRIBUTION

Probability distribution is the distribution which include two main parts.

i) Discrete probability distribution function.

ii) Continuous probability distribution function.

I. DISCRETE PROBABILITY DISTRIBUTION FUNCTION

This is the probability function (variable) which assumes separate value E.g $x_2 =$ 0, 1, 2, 3, 4…….

It consists of the following important parts;

i) Mathematical expectation

ii) Binomial probability distribution

iii) Poisson probability distribution

i) MATHEMATICAL EXPECTATION

Consider the values

$x_1, x_2, x_3, \ldots\ldots x_n$ with frequencies, $f_1, f_2, f_3 \ldots\ldots f_n$. respectively as shown below

X	X_1	X_2	X_3	……………………………	x_n
f	f_1	f_2	f_3	……………………………	f_n

From

$$Mean(\bar{x}) = \frac{x_1f_1 + x_2f_2 + x_3f_3 + \cdots + x_nf_n}{f_1 + f_2 + f_3 + \cdots + f_n}$$

$$\bar{x} = \frac{x_1f_1 + x_2f_2 + x_3f_3 + \cdots + x_nf_n}{\sum_{|z|}^{n} f_i}$$

$$\bar{x} = \frac{x1f1}{\sum_{|z|}^{n} fi} + \frac{x2fi2}{\sum_{|z|}^{n} fi} + \frac{x3f3}{\sum_{|z|}^{n} fi} + \ldots\ldots + \frac{xnfn}{\sum_{|z|}^{n} fi}$$

$\overline{x} = \frac{x1 . f1}{\sum_{|z|}^{n} fi} + \frac{x2 . fi2}{\sum_{|z|}^{n} fi} + \frac{x3 . f3}{\sum_{|z|}^{n} fi} + ... + \frac{xn . fn}{\sum_{|z|}^{n} fi}$

$$\overline{x} = x_1 p\ (x_1) + x^2\ p\ (x_2) + x_3\ p\ (x_3) + \ldots. + x_n\ p\ (xn)$$

$$\overline{x} = \sum_{i=1}^{n} xip\ (xi)$$

Therefore;

$$Mean(\bar{x}) = Expectation\ of\ x, E(x)$$

$$Mean(\bar{x}) = \sum_{i=1}^{n} xi. p(xi)$$

E (x) = $\sum_{i=1}^{n} xip\ (xi)$

Note:

1. E (x) = $\sum x\ p\ (x)$
2. $\sum\ p\ (x) = 1$

VARIANCE AND STANDARD DEVIATION

Variance

Recall

$$\text{Var}(x) = \frac{\sum f\,(x-\bar{x}^{2})}{\sum f}$$

$$= \frac{\sum F\,(x-\bar{x})(x-\bar{x}^{2})}{\sum F}$$

$$= \frac{\sum f\,(x^{2}-2x\,\bar{x}+\bar{x}^{2})}{\sum f}$$

$$= \frac{\sum fx^2}{\sum f} - \frac{\sum f2x\bar{x}}{\sum f} + \frac{\sum f\bar{x}^2}{\sum f}$$

$$= \frac{\sum fx^2}{\sum f} - \frac{2\bar{x}\sum fx}{\sum f} - \frac{\bar{x}^2\sum f}{\sum f}$$

$$= \frac{\sum fx^2}{\sum f} - \bar{x}^2$$

$$Variance = x^2 - (\bar{x})^2$$

But

$E(x) = \overline{x}$

$E(x^2) = \overline{x}^2$

$Var(x) = E(x^2) - [E(x)]^2$

Recall

Standard deviation, S.D = $\sqrt{var\ (x)}$

S.D = $\sqrt{E\ (x^2) -\ [E\ (x)]^2}$

Question

i) Given the probability distribution table

x	8	12	16	20	24
P (x)	1/8	1/6	3/8	1/4	1/12

Find i) E (x)

ii) E (x^2)

iii) $E(x - \bar{x})$

soln

Consider the distribution table below

x	8	12	16	20	24
$p(x)$	1/8	1/6	3/8	1/4	1/12
$x.p(x)$	1	2	6	5	2
x^2	64	144	256	400	576
$x^2.p(x)$	8	24	96	100	48
$x - \bar{x}$	-8	-4	0	4	8
$(x - \bar{x})^2$	64	16	0	16	64

i) $E(x) = \sum x.p(x)$

= 1 + 2 + 6 + 5 + 2

E(x) = 16

ii) $p(x) = \sum x^2.p(x)$

= 8 + 24 + 96 + 100 + 48

E (x^2)= 276

iii) $E(x - \bar{x})^2 = \sum (x - \bar{x})^2 px$

= 64(1/8) + 16 (1/16) + 0 (3/8) + 16 (1/4) + 64 (1/12)

$E(x-\bar{x})^2 = 18.33$

iv) $E(x-\bar{x}) = \sum (x-\bar{x})px$

= -8 (1/8) + -4 (1/16) + 0 (3/8) + 4 (1/4) + 8 (1/2

=1 – 2/3 + 1 + 2/3.

= 0

$E(x-\bar{x}) = 0$

Note

Always

$E(x-\bar{x}) = 0$

Proof

$E(x-\bar{x}) = E(x) - E(\bar{x})$

$= E(x) - \bar{x}E(1)$

$= \bar{x} - \bar{x}(1)$

$= 0$

= 0 proved

OR

$$E(x - \bar{x}) = \sum(x - \bar{x})p(x)$$

$$= \sum x\, p(x) - \bar{x}\sum p(x)$$

$$= \bar{x} - \bar{x}(1)$$

$$= 0$$

Prove

2) Given the distribution table

X	0	1	2
P (x)	K	2k	3k

Find (i) The value of k

(ii) $E(x - \bar{x})$

Soln

i) From the given table

$\sum p(x) = 1$

K + 2k + 3k = 1

6k = 1

K = 1/6

ii) The expected value

- Consider the distribution table below;

X	0	1	2
P (X)	1/6	1/3	1/2
X. PX	O	1/3	1

Hence

E (x) = $\sum x \cdot px$

= 0 + 1/3 + 1

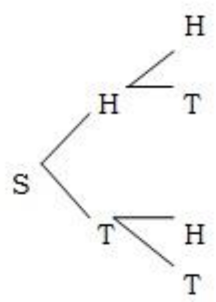
= 4/3

The expected value is E (x) = 4/3

ii) $E(x - \bar{x}) = 0$

3) In tossing a coin twice where x – represents the number of heads, appear, and construct the probability table for random experiment, form the table, calculate the expected value.

Tossing a coin twice



S = {HH, HT, TH, TT}

n (s) = 4

Probability distribution

X	0	1	2
P (x)	¼	½	¼
Xp (x)	0	½	½

X = 0

Hence

$$E(x) = \sum x.p(x)$$

= 0 + ½ + ½

E (x) = 1

The expectation of x is E (x) = 1

4. A class consists of 8 students. A committee of 4 students is to be selected from the class of which 4 are girls. If x – represent the number of girls, construct the probability table for random variable x and from the table, calculate the expected value.

- Consider the probability distribution below;

X	0	1	2	3	4
P (x)	1/70	16/70	36/70	16/70	1/70
Xp (x)	0	16/70	72/70	48/70	4/70

For x = 0

$$n(E) = 4C_0 . 4C_1 = 1$$

n (s) = $^8C_4$ = 70

P (E) = 1/70

For x = 1

n (E) = $^4C_1$. $^4C_3$ = 16

P (B) = 16/70

For x = 2

n (E) = $^4C_2 . ^4C_2 = 36$

P (B) = $\frac{n(E)}{n(s)} = \frac{36}{70}$

For x = 3

n (E) = $^4C_3 . ^4C_1 = 16$

P (E) = $\frac{n(E)}{n(s)} = \frac{16}{70}$

For x = 4

n (B) = $^4C_3 . ^4C_0 = 1$

p (E) = $\frac{n(E)}{n(s)}$

p (E) = 1/70

Hence

$\Sigma(x) = \Sigma x\, p\,(x)$

$$= 0 + \frac{16}{70} + \frac{72}{70} + \frac{48}{70} + \frac{4}{70}$$

= 2

The expectation of x is 2 E (x) = 2

05. Suppose a random variable x takes on value -3, -1, 2 and 5 with respectively probability $\frac{2x-3}{10}$, $\frac{x+1}{10}$, $\frac{x-1}{10}$ and $\frac{x-2}{10}$. Determine the expectation of x

From the given data

$\sum p(x) = 1$

$$\frac{2x-3}{10} + \frac{x+1}{10} + \frac{x-1}{10} + \frac{x-2}{10} = 1$$

2x – 3 + x + 1 + x – 1 + x – 2 = 10

5x – 5 = 10

5x = 15

X = 3

Consider the distribution table below;

X	-3	-1	2	5
P (x)	3/10	4/10	2/10	1/10
Xpx	-9/10	-4/10	4/10	

Hence,

$$E(x) = \sum x.p(x)$$

= -9/10 + -4/10 + 4/10 + 5/10

= -4/10

E (x) = -0.4

06. The random variable x has a probability distribution of 1/6 + 1/3 + 1/4.

Find the numerical values of x and y if E (x) = 14/3

Soln

From the given distribution table

$$\sum px = 1$$

1/6 + 1/3 + 1/4 + x + y = 1

x + y = 1 – 1/6 – 1/3 – ¼

x + y = ¼ ……..i

$$E(x) = \sum px$$

14/3 – 1/3 – 1 – 5/4 = 7x + 11y

25/12 = 7x + 11y

$7x + 11y = \frac{25}{12}$.....ii

Solving i and ii as follows;

$11\begin{cases} x + y = \frac{1}{4} \\ 7x + 11y = 25/12 \end{cases}$

$4x = \frac{11}{4} - \frac{25}{12}$

$X = \frac{1}{4}[\frac{11}{4} - \frac{25}{12}]$

X = 1/6

Also

X + y = ¼

1/6 + y = ¼

Y = ¼ = 1/6

Y = 1/12

The numerical values of x and y are such that x = 1/6, y = ½

07. A student estimates his chance of getting A in his subject is 10%, B+ is 40%, B is 35% C is 10%, D is 4% and E is 1%. By obtaining A , the students must get % points for B+, B, C, D and E, he must get 4, 3, 2, 1 and 0 respectively. Find the student's expectation and standard deviation.

Consider the distribution below;

	A	B^+	B	C	D	E
X	5	4	3	2	1	0
P (x)	0.1	0.4	0.35	0.1	0.04	0.01
X P (X)	O.5	1.6	1.05	0.2	0.04	0
X^2	25	16	9	4	1	0

From the table

E (x) = $\sum x\ px$

= 0.5 + 1.6 + 1.05 + 0.2 + 0.04 + 0

= 3.39

The student expectation is E (x) = 3. 39

Also

S.D = $\sqrt{E\ (x^2) - \ [E\ (x)]^2}$

$$E(x^2) = \sum x^2 px$$

= 2.5 + 6.4 + 3.15 + 0.4 + 0.04 + 0

= 12.49

S.D = $\sqrt{12.49-(3.39)^2}$

S.D = $\sqrt{0.9979}$

S.D = 0.99895

The standard deviation is 0.99895

THE EXPECTATION AND VARIANCE OF ANY FUNCTION

i) E (a) = a

Where a = is a constant

Proof

E (x) = $\sum x\ px$

$$E(x) = \sum x.p(x)$$

But

P (x) = 1

E (a) = a (s)

E (a) = a proved

ii) E (ax) = a E (x)

Where

a = constant

Proof

E (x) = $\sum x \, px$

E (ax) = $\sum ax \, p \, (x)$

$$E(ax) = a \sum xp(x)$$

But

$\sum x \, p \, (x)$ = E (x)

E (ax) = a E (x) proved

Where a and b are constant

Proof – 03

E (x) = Ex p (x)

E (ax + b) = $\sum(a + b)(p \, (x)$

= $\sum ax \, p \, (x) + \sum(ax + b)(p \, (x)$

$$= a \sum xp(x) + b \sum p(x)$$

= a E (x) + b (1)

E (ax + b) = a E (x) + b. Proved

4. Var (x) = 0

Where a – is constant
Proof 4

$$Var(x) = Ex^2 - [E(x)]^2$$

$a^2 - (a)^2$
$= a^2 - a^2$
= 0 proved

5. var (ax) = a^2 var (x)

Where

a – is any constant

Proof 05
Var (x) = E (x^2) – [E (ax)] ^{2}
var (ax) = E (ax) ^{2} – [E (ax)] ^{2}
= E (a^2 x^2) – [aE (x)] ^{2}
= a^2 E (x^2) – a^2 (E (x)] ^{2}
= a^2 E (x^2) – a^2 [E (x)] ^{2}
= a^2 [E (x^2) – a^2 [E (x)] ^{2}
= a^2 [E (x^2) – [E (x)] ^{2}]
var (x) = a^2 var (x)
Var (ax + b) = a^2var (x)
Where a and b are constant
Proof
Var (v) – E (ax + b) ^{2} – [E (ax + b)] ^{2}
Var (ax + b) = E (ax + b) ^{2} – [E (ax + b)] ^{2}
Var (ax + b) – E (a^2x^2 + 2abx + b^2) – [a Ex + b) ^{2}
= E (a^2x^2) + E (2abx) + E (b^2) – [a^2 E^2 (x) + 2ab E (x) + b^2
a^2 E (x^2) + 2abE (x) + b^2 – a^2 E^2 (x)

8. For random variable x show that var (x) – E (x^2) – [E (x)] ^{2}

b) The random variable has a probability density function p (x = x) for x = 1, 2, 3. As shown in the table below.

X	1	2	3
P (x)	0.1	0.6	0.3

Find i) E (5x + 3)

ii) E (x^2)

iii) var (5x + 3)

Consider the distribution table

X	1	2	3
P (x)	0.1	0.6	0.3
Xp (x)	0.1	1.2	0.9
X^2	1	4	9
x^2 Px	0.1	2.4	2.7

∑(5x + 3) = 5∑(x) + 3

But

$$\sum(x) = \sum x.px$$

= 0.1 + 1.2 + 0.9

= 2.2

$\Sigma(5x+3)$ = 5 (2.2) + 3

$\Sigma(5x+3)$ = 14

ii)

$$\sum(x^2) = \sum x^2px$$

= 0.1 + 2.4 + 2.7

$\Sigma(x^2)$= 5.2

iii) Var (5x + 3) = 5^2 var x

= 25 var (x)

Var (x) = $\Sigma(x^2)$ - $[\Sigma(x)]^2$

Var (x) = 5.2 – $(2.2)^2$

Var (x) = 0.36

Hence

Var (5x + 3) = 25 (0.36)

Var (5x + 3) = 9

Example

The discrete random variable x has probability distribution given in the table below;

Find var (2x + 3)

X	10	20	30
P (x)	01	0.6	0.3

From the given table

Var (2x + 3) = 2^2 var (x)

= 4 var x

But

$$Var(x) = \sum(x^2) - \left[\sum(x^2)\right]^2$$

Distribution table

X	10	20	30
P (x)	0.1	0.6	0.3
Xp (x)	1	12	9
X^2 p (x)	100	240	270
X^2	100	400	900

$$\sum(x) = \sum x.px$$

= 1 + 12 + 9

= 22

$$\sum(x^2) = \sum x^2 px$$

= 10 + 240 + 270

= 520

Var (x) = 520 – (22) ^{2}

Var (x) = 36

Therefore

Var (2x + 3) = 4 var (x)

= 4 (36) = 144

Var (2n + 3 = 144

BINOMIAL DISTRIBUTION

This is the distribution which consists of two probability values which can be distributed binomially

Properties

It has two probabilities, one is *probability of success* and one is a *probability of failure*.
The sum of probability of success p and of failure of q is one

P + q = 1

The trial must be independent to each other

It consist of n – number of trials of the experiment

Hence;

If p is the probability that an event will happen i.e (probability of success) and q is the probability that the event will not happen i.e (probability of failure) where n – is the number of trials

Then

The probability that an event occurs exactly x time from n – number of trials is given by

$P(x) = {}^{n}cx\ p^{x}\ q^{n-x}$

Where

n = is the number of trials

q = is the probability of failure

p = is the probability of success

x = is the variable

MEAN AND VOLUME , $\overline{X}$

Recall

$$Mean(\bar{x}) = \sum(x) = \sum x.px$$

$$\bar{x} = \sum x.px$$

$\bar{x} = \sum_{x=1}^{n} x^{n} cx\ p^{x} q^{n-x}$

Where

$x = 0, 1, 2, 3.....n$

$\bar{x} = 0^{n}C_{0}p^{0}q^{n-0} + 1^{n}C^{1}p\ q^{n-1}$

$+ 2^{n} C_{2} p^{2}q^{n-2} + 3^{n} C_{3}P^{3}q^{n-3}$

$+........n.^{n}C_{n} p^{n} q^{n-n}$

$= {}^{n}C_{1}pq^{n-1} + 2^{n} C_{2}p^{2}q^{n-2} +$

$3^{n} C_{3} p^{3} q^{n-3} +......+n^{n} C_{n} p^{n}q^{o}$

$= \frac{nb}{(n-1)b1b} pq^{n-1} + \frac{2nb}{(n-2)b\ 2b} p^{2}q^{n-2}$

$= \frac{n(n-1)b\ p\ q^{n-1}}{(n-1)\ b} + \frac{2n\ (n-1)(n-2)b}{(n-2)b\ 2b}$

$\frac{3n\ (n-1)(n-2)(n-3)b}{(n-3)\ b\ 3b} +.......+ np^{n}$

$= npq^{n-1} + n\ (n-1)\ p^{2qn-2} + \frac{n\ (n-1)(n-2)}{2b} p^{3}q^{n-3} + ...np^{n}$

$= np\ [^{qn-1} + \frac{(n-1)\ p\ q^{n-2}}{1b} + \frac{(n-1)\ (n-2)\ p^{3} q^{n-3} + \cdots np^{n}}{2b}$

$= np\ [^{qn-1} + \frac{(n-1)\ p\ q^{n-2}}{1b} + \frac{(n-1)\ (n-2)\ p^{2} q^{n-3} + \cdots p^{n-1}}{2b}$

$= np\ (p + q)^{n-1}$

But

P + q = 1

$= np\,(1)^{n-1}$

= np

$$\bar{x} = np$$

VARIANCE

$$Var(x) = E(x^2) - [E(x)]^2$$

Taking

$$E(x^2)$$

$\sum(x^2) = \sum[x\,(x-1) + x]$

$= \sum x\,(x-1) + \sum x$

$\sum(x\,(x-1)) = \sum x\,(x-1)p\,(x)$

$$= \sum_{x=i}^{n} x(x-1)n_{C_x}\,P^x q^{n-1}$$

$$i = 1,2,3,4$$

$= O^n C_1 p^1 q^{n-1} + 2^n C_2 P^2 q^{n-2} + 6^n C_3 P^2 q^{n-3} \ldots + n\,(n-1)\,{}^nC_n p^n q^n$

$= 2^nC_2p^2q^{n-2} + 6n\,C_3\,p^3q^{\,n-3} + ... + n\,(n-1)\,{}^nC_n\,P^nq^0$

$= \frac{2nb}{(n-2)b\;2b}\,p^2q^{n-2} + \frac{6nb\,p^{n-3}}{(n-3)b\;3\;b} + ... + n\,\frac{n\,(n-1)nb\;p^n}{(n-n)\,b\;n\;b}$

$= \frac{n\,(n-1)\,(n-2)b\;p^2\,q^{n-2}}{(n-2)b} + \frac{n\,(n-1)\,(n-2)\,(n-3)b}{(n-3)b\;1b} + ... + n\,(n-1)\,p^n$

$= n\,(n-1)\,p^2q^{n-2} + \frac{n\,(n-1)(n-2)\,p^3\,q^{n-3}}{1b} + n\,(n-1)\,p^n$

$= n\,(n-1)\,p^2\,[q^{n-2} + \frac{(n-2)p\;q^{n-3} + ... + p^{n-2}}{1b}]$

$= n\,(n-1)\,p^2\,(p+q)^{\,n-2}$

$= p\,(n-1)\,p^2$

Hence

$\sum(x^2) = n\,(n-1)\,p^2 + np$

$= np\,(n-1)\,p + 1)$

$$Var(x) = \sum(x^2) - \left[\sum(x^2)\right]^2$$

$= np\,(n-1)\,p + 1) - (np)^2$

$= np\,[(n-1)\,p + 1 - np]$

$= np\,[np - p + 1 - np]$

$= np\,[1 - p]$

$= npq$

Var (x) = npq

STANDARD DEVIATION

From S.D = $\sqrt{var\ (x)}$

S.D = $\sqrt{npq}$

Note

From p (x) = nCx $p^x q^{n-x}$

i) $n \le 50$

ii) $\Sigma(x)$ = np

iii) var (x) = npq

iv) S.D = $\sqrt{npq}$

Question

1. A pair coin is tossed 12 times the probability of obtaining head is 0.5, determine mean and standard deviation.

2. If x is a random variable such than $\Sigma(x) = 2.4$ and p = 0.3

Find the value of n and S.D

3. Suppose that, the rain office records. Show that averages of 5 days in 30 days in June are rainy days. Find the probability that June will have exactly 3 rainy days by using binomial distribution also find S.D.

POISSON DISTRIBUTION

This is the special case of binomial probability distribution when the value of *n is very large number (n > 50)* and when the *probability of success, p is very small* i.e ($p < 0.1$)

Properties

The Condition for application of poison probabilities distribution are

i) The variable x must be discrete random

ii) The occurrence must be independent

iii) The value of n is always greater than 50 (i.e n) 50) and the probability of success, p is very small i.e $p<0.1$

iv)

$$Mean(\bar{x}) = \sum x = np = x$$

$$x = np$$

Therefore;

$P(x) = {}^{n}Cx\, p^{x}q^{n-x}$

But $p + q = 1$

$q = 1 - p$

$$p(x) = n_{C_x}\left(\frac{x}{n}\right)^x (1 - x/n)^{n-x}$$

$$= \frac{nb}{(n-x)bxb}(x/n)^x(1 - x/n)^{n-x}$$

$$= \frac{n\,(n-1)(n-2)\,(n-3)\ldots\ldots(n+1-x)(n-x)b\,(x/n)^x\;(1-\lambda/n)\;n-x}{xb}$$

$$= \frac{\frac{n}{n}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\left(\frac{n-3}{n}\right)\ldots\ldots\left(\frac{n+1}{n}-\frac{n}{x}\right)x^x\;(1-\lambda/n)^{n-x}}{xb}$$

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)(1-\left(\frac{x-1}{n}\right)x^x\;(1-\frac{x}{n})^n}{xb\;(1-x/n)^n}$$

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\ldots\ldots\ldots(1-\left(\frac{x-1}{n}\right)x^x\;[(1-\frac{x}{n})^n/x]}{xb\;(1-\lambda/n)^n}$$

But

$$(1 - x/n)^{-n/x} = e$$

$$Ans\; n \rightarrow \propto$$

$$\text{P (x)} = \frac{1\,(1-0)(1-0)\,(1-0)\ldots\ldots(1-0)x^x\;(e)^{-x}}{xb\;(1-0)^x}$$

$$\text{P (x)} = \frac{x^x e^{-\lambda}}{xb}$$

Note;

$(1 + 1/x)^x = e$

X = ∞

MEAN AND VARIANCE

$$Mean(\bar{x})$$

From $\Sigma(x) = \Sigma x\, p\,(x)$

$= \Sigma \frac{x\, \lambda^{x} e^{-x}}{x b}$

$$= \sum_{x=1}^{n} \frac{x(x^{x})(e^{-x})}{x(x-1)b}$$

$$= \sum_{x=1}^{n} \frac{x.x^{x}e^{-x}}{x(x-1)b}$$

Where

$$i = 1,2,3,4...$$

$$= x^{2}e^{-x}\left[\frac{x^{0}}{0b} + \frac{x^{1}}{1b} + \frac{x^{2}}{2b} + \frac{x^{3}}{3b} + \frac{x^{x-2}}{(n-2)b}\right]$$

$$= x^{e^{-x}.e^{x}}$$

$$= x^{e^{0}}$$

$$= x$$

Therefore

$Mean(\bar{x}) = x$

= np

Variance

Var (x) = $\Sigma(x^2)$ – E^2 (x)

Taking

$\Sigma(x^2)$ = n $\Sigma(x^2)$

But

$\Sigma(x^2)$ = $\Sigma(x\ (x-1)+x)$

= $\Sigma\, x(x-1)$ + $\Sigma(x)$

Taking

$$\Sigma\, x(x-1) = \Sigma\, x\,(x-1)px$$

$$= \Sigma_{x=1}^{n} x\,(x-1) \cdot x^{X} e^{-X}/x\,(x-1)\,(x-2)$$

$$= \sum_{k=0}^{n} \frac{x^{x} e{-x}}{(x-2)\,b}$$

$$\sum_{k=0}^{n} \frac{x^{x} e^{-x}}{(x-2)b}$$

$$x^{2} e^{-x} \sum_{x=0}^{n} \frac{x^{x-2}}{(x-2)b}$$

Where;

i = 2, 3, 4

$$= x^2 e^{-x} \left[\frac{x^0}{0b} + \frac{x^1}{1b} + \frac{x^2}{2b} + \frac{x^3}{3b} + \frac{x^{x-2}}{(n-2)b}\right]$$

$= x^2 e^{-x} . e^x$

$= x^2$

Hence,

$\sum x (x - 1)$

$\sum (x^2) = \sum x (x - 1) + \sum(x)$

$$= x^2 + x$$

Therefore;

Var (x) = $\sum(x^2) - \sum^2 (x)$

$$= x^2 + x - x$$

$$Var(x) = x$$

Var (x) = np

STANDARD DEVIATION

From

S.D = $\sqrt{var\ (x)}$

= $\sqrt{np}$

S.D = $\sqrt{np}$

Question

1. Given that probability that an individual is suffering from moralia is 0.001. Determine the probability that out of 2000 individual

i) Exactly 3 will suffer

ii) At least 2 will suffer

2. Use poison distribution, find the probability that a random sample of 8000 people contain at most 3 NCCR members if an average 1 person in each 1000 members is NCCR member.

3. Random variable x for a poison distribution, if

P (x = 1) = 0.01487

P (x = 2) = 0.0446. Find

P (x = 3)

b) Find the probability that at most 5 defective fuses will be found in a box of 200 fuses of an experience shows that 2% of such fuses are defective.

B. CONTINUOUS PROBABILITY DISTRIBUTION

These are two parts of continuous probability distribution, these are

i) The variable x must be continuous

ii) The function is integrable

iii) The area under the curve are

$= \int_{-\propto}^{\propto} p\ (x)dx$

iv) For the curve f (x)

i.e f (x) > 0 or f (x) ≥1

RULES

i) $P(x = 0) = 0\ for - \propto < x < 1$

ii) $P(a \leq x \leq b \leq) = P(a < x < b)$

$= \int_{a}^{b} p\ (x)dx$

iii) P (x < x) = $\int_{-\propto}^{\propto} p\ (x)dx$

For instance

$P(x < 0.6) = \int_{\propto}^{0.6}$

iv) P (x > x) = $\int_{x}^{\infty} p\,(x)dx$

For instance

P (x > 0.2) = $\int_{0.2}^{\infty} p\,(x)dx$

· **Note**

The sufficient conditions for p (x) to be continuous distribution are

i) P (x > o) at (a, b)

ii) Area under the curve is 1 i.e $\int_{a}^{b} p\,(x)dx = 1$

Mean variance by probability density function (p.d.f)

$$Mean(\bar{x}) = \sum(x) = \int_{a}^{b} xp(x)dx$$

$$Mean(\bar{x}) = \int_{a}^{b} xP(x)dx$$

Variance

From

$$\text{Var (x)} = \int_{a}^{b}(x - \overline{x})^2 p\,(x)dx$$

$$= \int_{a}^{b} p\,(x^2 - 2x\,\overline{x} + \overline{x}^2)\text{p(x) dx}$$

$= \int_a^b x^2\, p\,(x)\, dx - \int_a^b 2x\,\bar{x}\, p\,(x) dx + \int_a^b \overline{x^2} p\,(x) dx$

$= \int_a^b x^2\, p\,(x) dx - 2\bar{x} \int_a^b x\, p\,(x) dx + \overline{x^2} \int_a^b p\,(x) dx$

$= \int_a^b x^2\, p\,(x) - 2\bar{x}\,\bar{x} + \overline{x^2}(1)$

$= \int_a^b x^2\, p\,(x) dx - x^2$

$= \int_a^b x^2\, p\,(x) dx - [\int_a^b x\, p\,(x) dx\,]^2$

Therefore

$$Var(x) == \int_a^b x^2\, p(x) dx - [\int_a^b xp\, dx]^2$$

Example

A continuous random variable x has a probability function given by

$$P\,(x) = \begin{cases} ax - bx^2, 0 \le x \le 2 \\ 0,\ else\ where \end{cases}$$

Observation in x indicates that expectation of x is 1, show that a = 1.5 and find value of b

Solution

P (x) = ax - bX^2, 0 ≤ x ≤ 2

$$P(x) = 0, -\propto \le x \le \propto$$

Also

$\int_a^b p\,(x)dx = 1$

= $\int_0^2 ax - bx^2$ + $\int_{-\propto}^{\propto} 0$ dx = 1

= $\int_0^2 ax - bx^2 + \int_{\propto}^{\propto} 0$ dx = 1

= $\int_0^2 ax$ - $\int_0^2 bx^2$ + 0 =1

Note

$\int_b^a odx = 0$

= $\int_0^2 ax\ dx$ - = $\int_0^2 bx^2 = 1$

= a$\int_0^2 x\ dx$ - b $\int_0^2 x^2$ = 1

$$a\left[\frac{x^2}{2}\right]_0^2 - b\left[\frac{x^3}{3}\right]_0^2 = 1$$

$$2a - \frac{8}{3}b = 1$$

Also

Æ© (x) = $\int_a^b p\,(x)dx$

1 = $\int_0^2 x\,(ax - bx^2)dx + \int_{-\propto}^{\propto} x\,(o)$ dx

1 = $\int_a^b (ax^2 - bx^3)dx$

$$a\left[\frac{x^3}{3}\right]_0^2 - b\left[\frac{x^4}{4}\right]_0^2 = 1$$

$$\frac{8}{3}a - 4b = 1$$

$$8a - 12b = 3$$

6a -8b = 3
2a – 4b = 0
a = 2b
b = a/2
8 (a) – 12 (a/2) = 3
8a – 6a = 3
2a = 3
a = 1.5 shown

Also

$$\frac{a}{2} = b$$

$$\frac{1.5}{2} = b$$

b = 0.75

Example

The random variable x denotes that the number of weeks of a certain type of half life of the probability density function f (x) is given by

$$f(x)\begin{cases} 200/x^2 & x > 100 \\ o\ else\ where \end{cases}$$

Find the expected life

soln

From

Æ© (x) = $\int x\, p\,(x)dx$

$\Sigma(x) = \int_{100}^{\infty} x.\frac{200}{x^3} + \int_{\infty}^{\infty} x.0$

$= \int_{100}^{\infty} \frac{200}{x^2}$

$= \int_{100}^{\infty} \frac{200}{x^2}$ dx

= 200 $\int_{100}^{\infty} \frac{1}{x^2}$ dx

= 200 $[\frac{-1}{x}]_{100}^{\infty}$

= 200 $[\frac{1}{100} - 0]$

= 2

Σx = 2 weeks

$= \int_a^b (x^2 - 2x\bar{x} + x^2) p\,(x)dx$

$$= \int_a^b x^2\, p(x)dx - \int_a^b 2x\,\bar{x}\, p(x)dx + \int_a^b \bar{x}^2\, p(x)dx$$

$$= \int_a^b x^2\, p(x)dx - 2x \int_a^b x\,(x)dx + \bar{x}^2 \int_a^b p(x)\, dx$$

$$= \int_a^b x^2\, p(x)dx - 2x\bar{x} + x^2$$

= $\int_a^b x^2$ p (x) dx – x

= $\int_a^b x^2$ p (x) dx – [$\int_a^b x\, p\,(x)dx]^2$

Therefore

$$Var(x) = \int_a^b x^2\, p(x)dx - \bar{x}^2$$

Example

A continuous random variable x has a probability function given by

P (x) = $\begin{cases} ax - bx^2, O \leq x \leq 2 \\ O,\ else\ where \end{cases}$

Observation in x indicates that expectation of x is 1, show that a = 1.5 and find value of b

Soln

P (x) = ax - bx^2, O $\leq$ X ≤ 2

P (x) = 0, -$\propto \leq$ x $\leq \propto$

Also

$\int_a^b p\,(x)dx = 1$

= $\int_o^2 ax - bx^2$ + $\int_\propto^\propto 0$ dx = 1

= $\int_a^b ax$ - $\int_a^b bx^2$ + 0 = 1

Note

$$\int_b^a 0dx = 0$$

$\int_0^2 ax\,dx$ - $\int_0^2 bx^2$

$$= a\int_0^2 x\,dx - \int_0^2 bx^2\,dx = 1$$

$$= a\left[\frac{x^2}{2}\right]_0^2 - b\left[\frac{x^3}{3}\right]_0^2 = 1$$

$$= 2a - \frac{8b}{3} = 1$$

Also

$\Sigma(x)$ = $\int_a^b xp\,(x)dx$

Example

Given that the probability distribution function for random variable x is given by

$$f(x) = \begin{cases} 1x1;\ 0 \le x \le 2 \\ 0, else\ where \end{cases}$$

Find the expected value

Solution

$f(x) = 1x1, \quad 0 \le x \le 2$

$f(x) = 0, \ -\propto \le x \le \propto$

But

$1x1 = \pm x$

$1x1 = x \; if \;\; x > 0$

$1x1 = -x \; if \;\; x < 0$

Now

$= \int_0^2 x|x|dx + \int_{-\alpha}^{\alpha} x.0$

$= \int_0^2 x^2 dx + \int_{\alpha}^{\alpha} o dx$

$= \left[\frac{x^3}{3}\right]_0^2$

$= {}^8/_3 - 0$

$= {}^8/_3$

Expected value is ${}^8/_3$

Example

A function

$$f(x) = \begin{cases} 0, x < 2 \\ {}^1/_{18}(cx + 3), 2 \leq x \leq 4 \\ 0, 4 < x \end{cases}$$

Find the value of c if it is a probability density function hence calculate

(i) Mean

(ii) Variance

Solution

$f(x) = 0, x < 8$

$$f(x) = \frac{1}{8}(cx + 3) for\ 2 \leq x \leq 4$$

$f(x) = 0, \qquad 4 < x$

$1 = \int_{-\propto}^{2} odx + \int_{2}^{4} \frac{1}{18}(cx + 3) + \int_{4}^{\propto} odx$

$\frac{1}{18}\int_{2}^{4}(cx + 3) = 1$

$\frac{1}{18}\left[\frac{cx^2}{2} + 3x\right]_2^4 = 1$

$\frac{1}{18}\left[\frac{16c}{2} + 12\right] - \left[\frac{4c}{2} + 6\right] = 1.$

$8c + 12 - [2c + 6] = 18$

$8c + 12 - 2c - 6 = 18$

$6c + 6 = 18$

$6c = 12$

$c = 2$

(i)Mean

$\Sigma(x) = \int_{b}^{a} x\ p(x)dx$

$= \int_{-\alpha}^{2} x.o\, dx + \int_{2}^{4} x\left(\frac{1}{18}(2x+3)\right) dx$

$= \int_{4}^{\alpha} x.o\, dx$

$= \int_{2}^{4} \frac{1}{18}[2x^2 + 3x]$

$= \frac{1}{18}\int_{2}^{4} 2x^2 + 3x$

$= \frac{1}{18}\left[\frac{2x^3}{3} + \frac{3x^3}{2}\right]_2^4$

$= \frac{1}{18}\left[\frac{2x64}{3} + \frac{3}{2}.16\right] - \left[\frac{2.8}{3} + \frac{3.4}{3}\right]$

$= \frac{1}{18}\left[\frac{128}{3} + 24\right] - \left[\frac{16}{3} + 6\right]$

$= \frac{1}{18}\left[\left(\frac{128}{3} + 24 - \frac{16}{3} - 6\right)\right]$

$= \frac{1}{18}\left[\frac{112}{3} + 18\right]$

$= \frac{112}{54} + 1$

= **3.074**

(ii) Var(x)

From

$$Var(x) = \int x^2 p(x)dx - \left[\int x\, p(x)dx\right]^2$$

$$= \int_{\propto}^{2} x^2.0 + \int_{2}^{4} x^2.{}^{1}/_{18}(2x+3)dx$$

$$= -\int_{\propto}^{4} x^2 odx$$

$$= \int_{2}^{4} x^2\left(\frac{1}{18}(2x+3)\right)dx - (3.074)^2$$

$$= \frac{1}{18}\int_{2}^{4} 2x^2 + 3x^2 - (3.0+4)^2$$

$$= \frac{1}{18}\left[\frac{2x^4}{4} + \frac{3x^3}{3}\right]_2^4 - (3.074)^2$$

$$= \frac{1}{18}\left[\left[\frac{2x256}{4} + \frac{3x64}{3}\right] - \right]\left(\frac{2x16}{4} + \frac{3x3}{4}\right)$$

$= -(3.074)^2$

$= \frac{1}{18}[(128 + 64) - (8 - 8) + (3.074)^4]$

$= 9.7778 - (3.074)^2$

$= 0.3285$

NORMAL DISTRIBUTION

Normal distribution is a continuous distribution.

It is derived as the limiting form of binomial distribution for the large values of n where p and q are not very large.

$$f(x) = \sigma \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-ee)^2}{2\sigma^2}} dx$$

$$Where\ ee = Mean(\bar{x})$$

$$\sigma = standard\ \ deviation$$

$$\pi = 3.14$$

$$e = 2.7183$$

$$p(a < x < b) = \int_a^b \sigma \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-ee)^2}{2\sigma^2}} dx$$

STANDARD VALUE

For standard value

$$z = \frac{x-ee}{\sigma}$$

Where

$\check{z} = standard\ value\ (score)$

X = variable

$\sigma = standard\ detation$

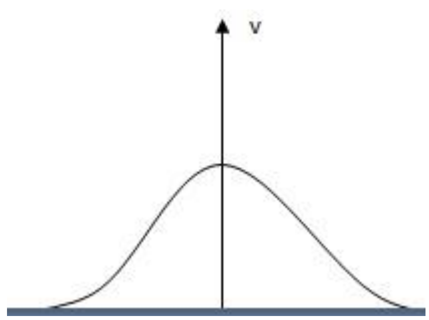
$Where\ ee = Mean(\bar{x})$

Hence

$f(\check{z}) = \sigma \frac{1}{\sqrt{2\pi}} e^{\ \frac{-1}{2} 7^2}$

NORMAL CURVE

A frequency diagram can take a variety of different shapes however one particular shape occurs in many circumstances



-This kind of diagram is called NORMAL CURVE

PROPERTIES OF NORMAL

DISTRIBUTION CURVE

(i) The curve is symmetrical about the mean

$$(x) = ee \ where \ z = 0$$

(ii) The value of

$$x = z \ range \ from -\propto \ to \ \propto$$

(iii) As $(x = z) \Rightarrow \pm\propto$

$$f(x = z) => 0$$

(iv) The curve never to.......x-................

(v) The curve is maximum at x = ee

(vi)The area under the curve is one re area (A) = 1 square unit

AREA UNDER NORMAL CURVE

By taking

$$z = \frac{x-ee}{\sigma}$$

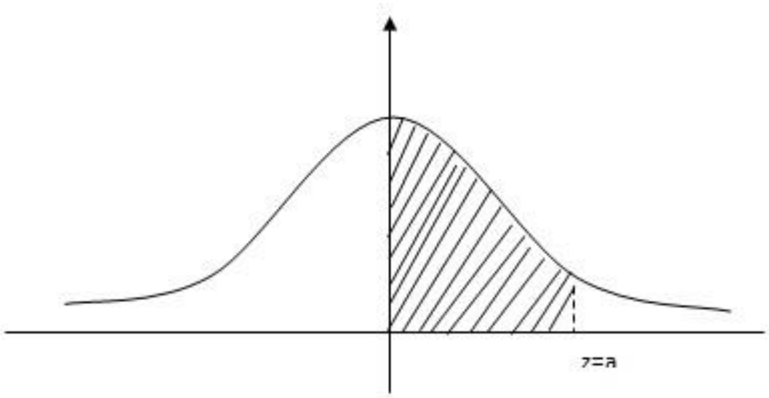
The standard normal curve is found.

- The total area under the curve is one.
- The area under the curve is divided into two equal parts by zero.
- The left hand side area is 0.5 and the right hand side area is 0.5

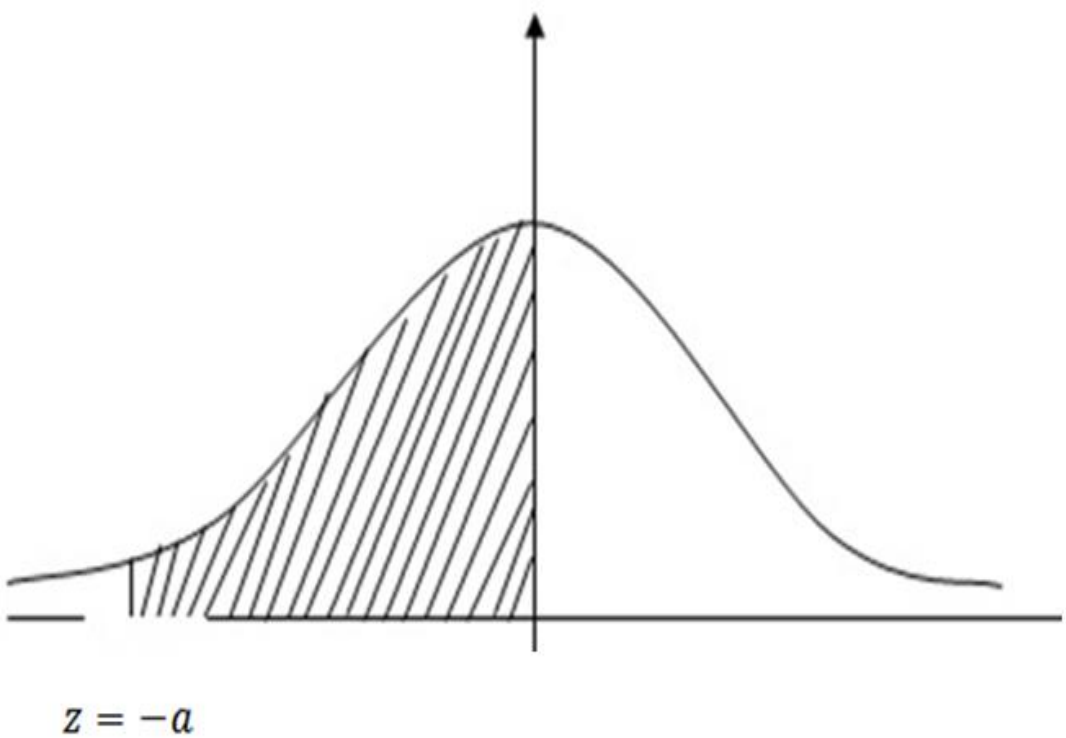
- The area between the ordinate $z = 0$ and any other ordinate can be noted from the TABLE or CALCULATOR

Probability from Normal distribution curve.

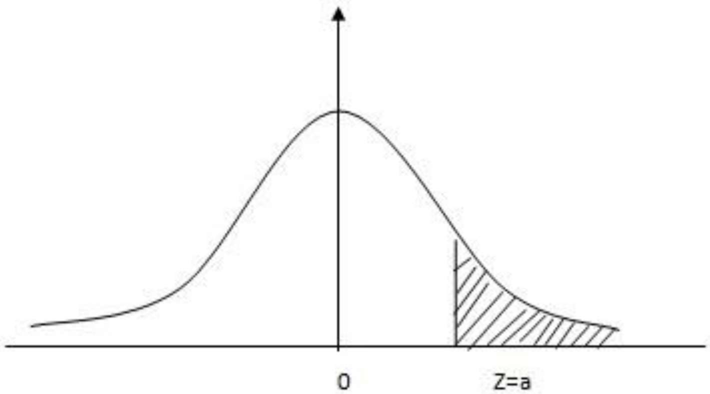
1. $p(z = a)$



2. $p(z = -a)$

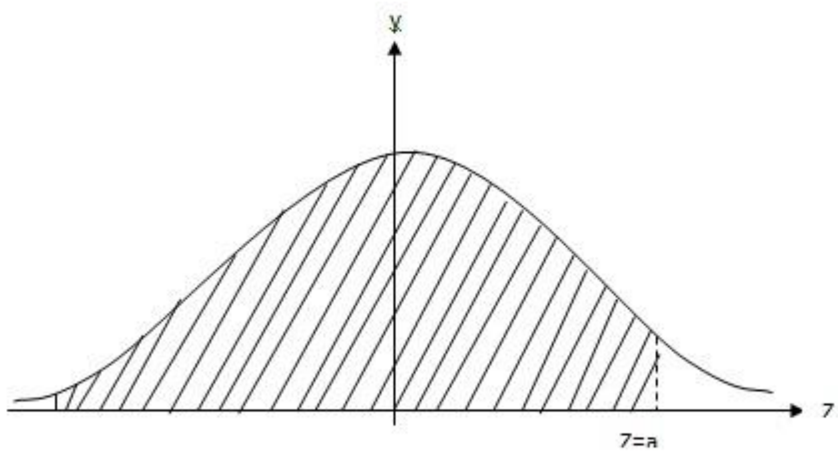


3. $p(z \geq a)$



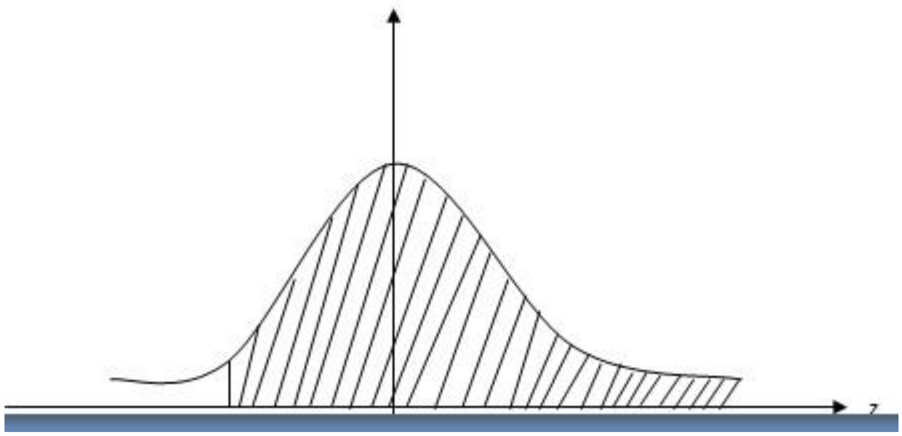
Note: $p(z \geq a)$=0.5 - $p(z = a)$

4. $p(z \leq a)$



NOTE: $p(z \leq a) = 0.5 + p(z = a)$

5. $p(z \geq -a)$

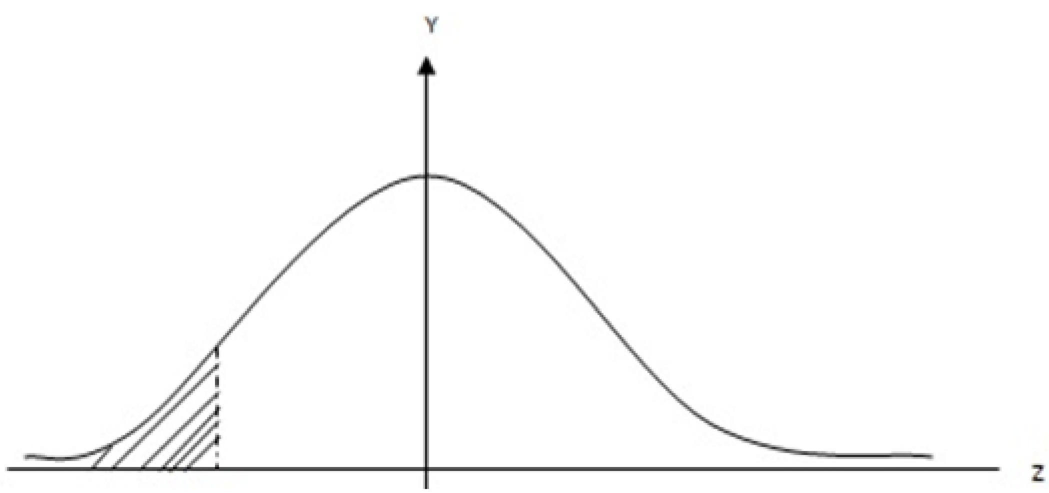


Note:

$p(z \geq -a) = p(z = -a) + 0.5$

$= p(z = a) + 0.5$

6. $p(z \leq -a)$



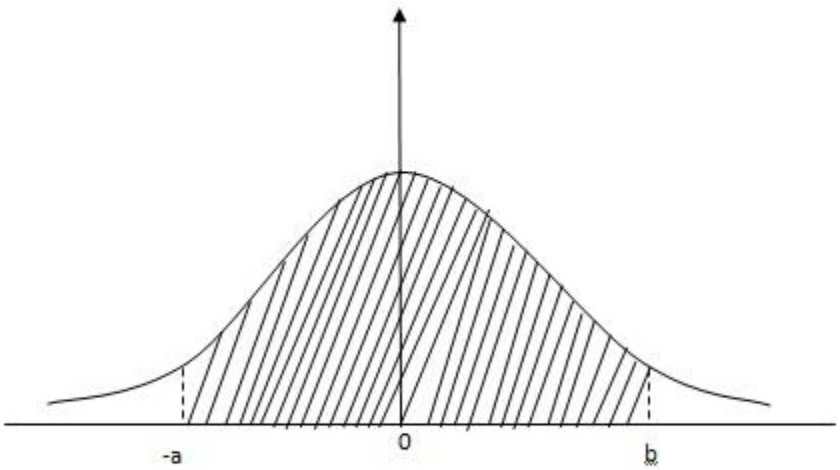
$z = -a$

Note:

$p(z \leq -a) = 0.5 - p(z = -a)$

$0.5 - p(z = a)$

7. $p(-a \leq z \leq b)$

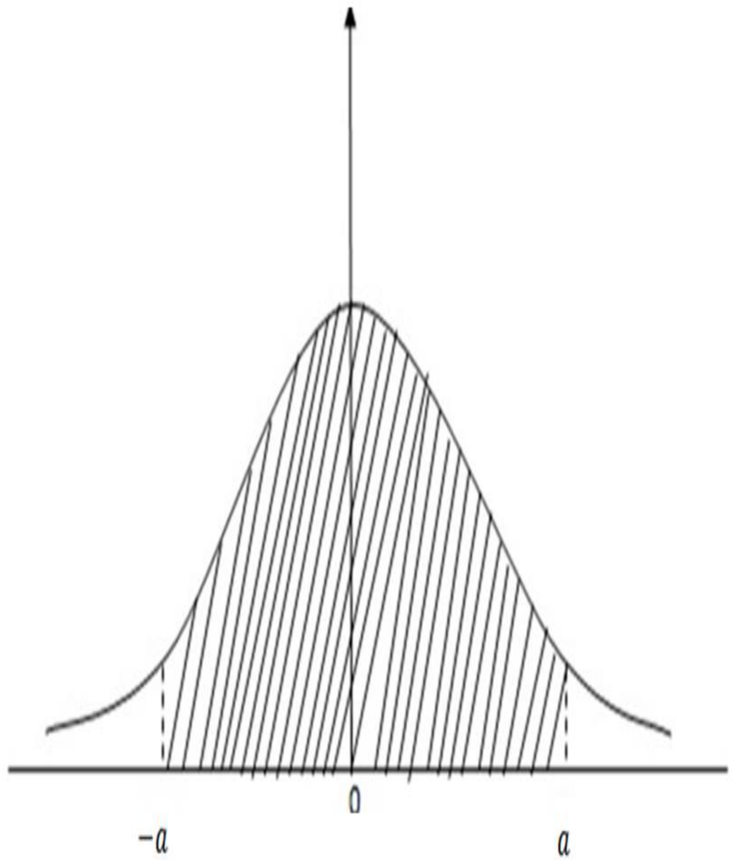


Note:

$p(-a \leq z \leq b) = p(z = -a) + p(z = b)$

$= p(z = a) + p(z = b)$

8. $p(-a \leq z \leq a)$



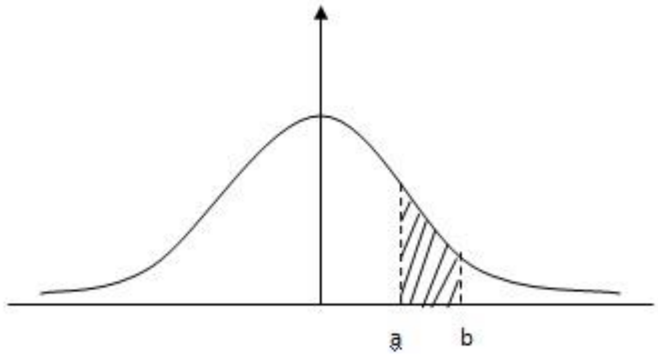
Note:

$p(-a \leq z \leq a) = p(z = a) + p(z = a)$

$= p(z = a) + p(z = a)$

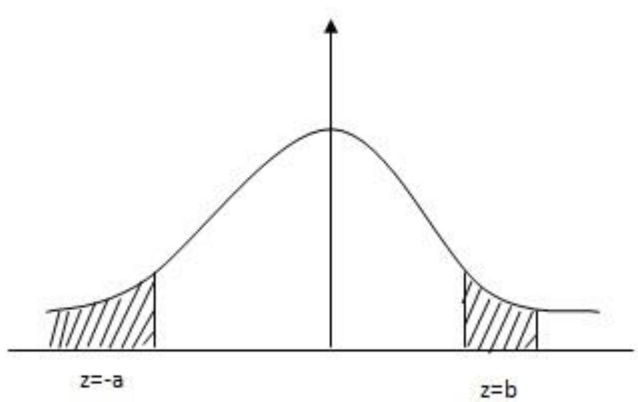
$= 2. p(z = a)$

9. $p(a \leq z \leq b)$



$p(a \leq z \leq b) = p(z = b) - p(z = a)$

10. $p(-a \geq z \text{ or } y \leq z)$



Note:

$p(-a \geq ? \; or \; b \leq ?) = 0.5 - (? = -a) + 0.5 - p(? = b)$

$= 1 - [p(? = a) + p(? = b)]$

STATISTICAL CALCULATION

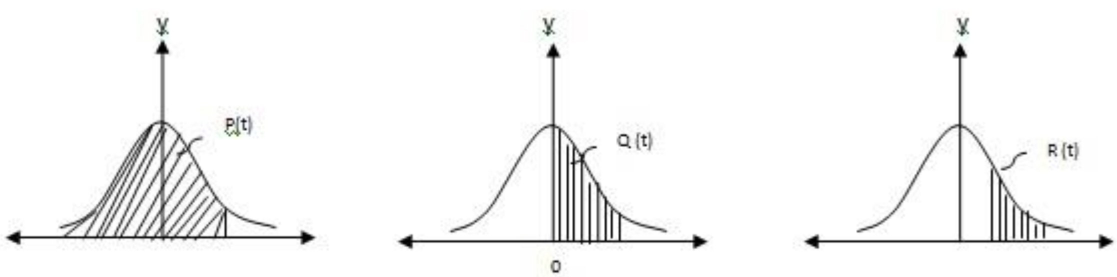
(NORMAL DISTRIBUTION)

Consider the set up screen shown below;

Small between large

→ PC QC RC t

1 2 3 4



Therefore

$X \quad t \quad = x - \bar{x}$

xQn

where

$x \; = \; variable$

$\bar{x} = ee = mean$

$xQn \; = (\sigma) = standard \; deviation$

$x \rightarrow t = z = Normalized\ vante$ (standard score)

Question

Find the area under the normal curve in each of the following cases;

(a (a) $z = 0\ and\ z = 1.2$

(b (b) $z = -0.68\ and\ z = 0$

(c (c) $z = -0.46\ and\ z = 2.21$

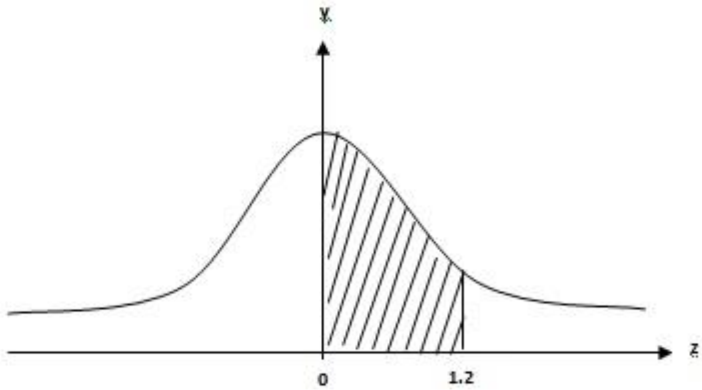
(d (d) $z = 0.81\ and\ z = 1.94$

(e (e) $z \leq 0.6$

(f) $z \geq -1.28$

Solution (a)

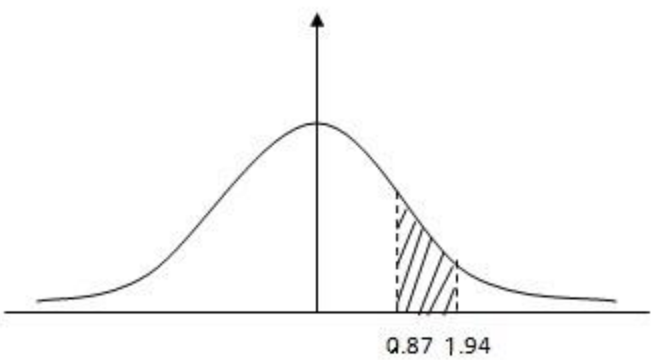
$z = 0\ and\ z = 1.2$



$p(z = 1.2) = \emptyset(1.2)$

Area = 0.3849sq unit

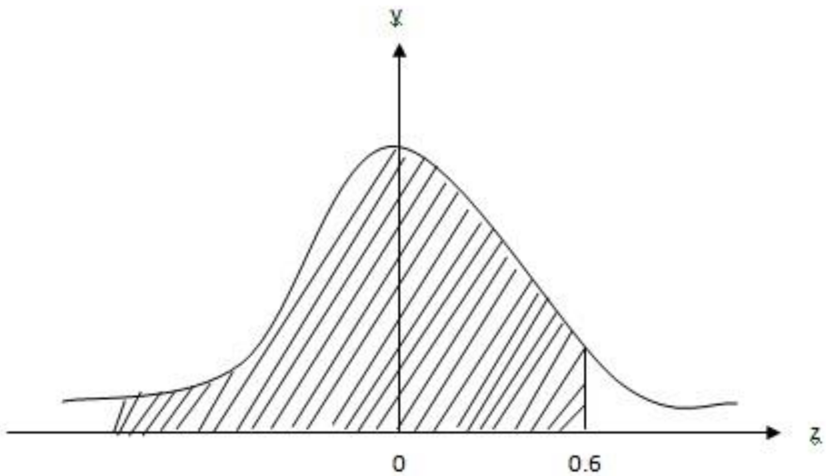
(d) $z = 0.87 \; and \; z = 1.94$



$p(0.87 \leq z \leq 1.94) = \emptyset(1.94) - \emptyset 0.84$

$= \; 0.1828 \; sq \; unit$

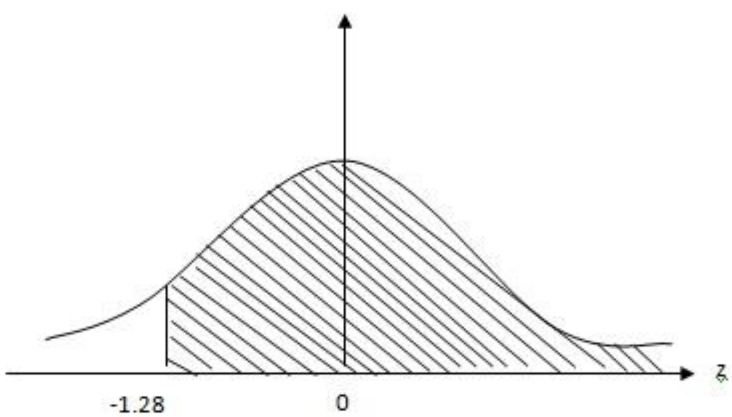
(e) $z \leq 0.6$



$p(z \leq 0.6) = 0.5 + \emptyset(0.6)$

= 0.7258

(f) $z \geq -1.28$



$p\left(z \geq -1.28 = \emptyset(-1.28)\right)$

$= 0.89973\ sq\ unit$

Question

Determine the normalized variety (→t)p(t) for x=53 and normal distributions

P(t) for the following data 55, 54, 51,55, 53, 53, 54, 52

Solution

$\bar{x} = 53.375$

$x\sigma n = 1.317$

From

$\partial = \frac{x-\bar{x}}{x\sigma n}$

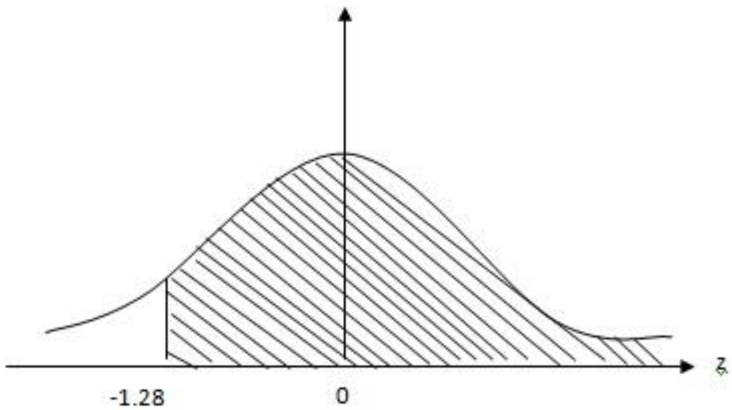
$x \rightarrow$

By using scientific calculation

53 – t

$\partial = -0.284747398$

= -0.28



$p(z \leq -0.28) = \emptyset(-0.28)$

= 0.38974

Question

The marks in Mathematics examination are found to have approximately normal distribution with mean 56 and standard deviation of 18. Find the standard mark equivalent of a mark 70.

Solution

$\bar{x} = 56$

$x\sigma n = 18$

$z = ?$

$z = \frac{x-\bar{x}}{x\sigma n}$

$z = \frac{70-56}{18}$

$z = 0.7778$

= 0.78

$\frac{0.78}{100} x100 = \frac{78}{100} x\ 100\%$

= 78%

The standard mark equivalent to a mark of 70 is 78%

Question

Assuming marks are normally distributed with means 100 and standard deviation 15. Calculate the proportional of people with marks between 80 and 118

Solution

$\bar{x} = 100$

$x\sigma n = 15$

$p(88 \leq x \leq 118)$

But

$z = \frac{x-\bar{x}}{x\sigma n}$

$z_1 = \frac{88-100}{1\sigma}$

= -0.8

$z_2 = \frac{118-100}{1\sigma}$

= 1.2

$p(-0.8 \leq z \leq 1.2)$

$P(-0.8 \leq z \leq 1.2) = \emptyset(1.2) + \emptyset(0.8)$

$= 0.6733$

$= 67.31\%$

The proportional of the people with marks between 88 and 118 is 67.31%

Question

(a State the properties of normal distribution curve

(b Neema and Rehema received standard score of 0.8 and 0.4 respectively in Mathematics examination of their marks where 88 and 64 respectively. Find mean and standard deviation of examination marks.

$z_1 = 0.8$

$z_2 = 0.4$

$x_1 = 0.8$

$x_2 = 0.4$

From

$2 = \frac{x-\bar{x}}{x\sigma n}$

$0.8 = \frac{88-\bar{x}}{x\sigma n}$

$0.8\ x\sigma n = 88 - \bar{x}$

$0.8\ x\sigma n + \bar{x} = 88 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (i)$

$-0.4 = \frac{64-\bar{x}}{x\sigma n}$

$-0.4x\sigma n = 64 - \bar{x}$

$-0.4x\sigma n + \bar{x} = 64 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (ii)$

$\bar{x} + 0.8x\sigma n = 88$

$\bar{x} - 0.4x\sigma n = 64$

$\bar{x} = 72$

$x\sigma n = 20$

THE NORMAL APPROXIMATION (N) TO THE BINOMIAL DISTRIBUTION (B)

Suppose x is the discrete variety distributed as $B(n, p)$. then this can be approximately $N(np, npq)$ if and only if

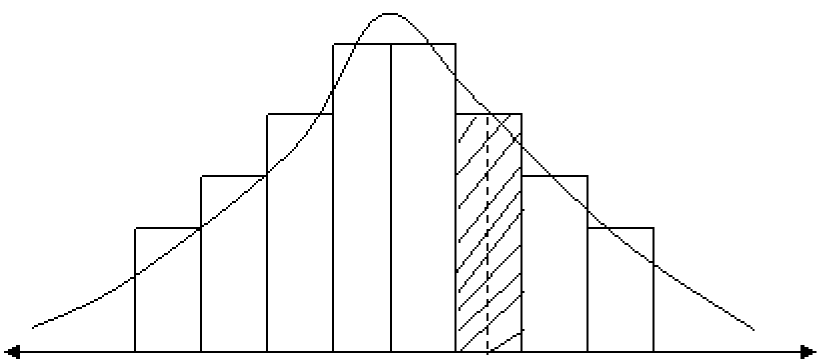
(i) $n > 50$

(ii) P is not too large or too small re $(0.2 \leq P \leq 0.8)$

Note:

(i)$the\ notation\ for\ normal\ distribution\ is\ N(ee, \delta^2)$

(ii)



(A normal approximation to binomial distribution)

For x considered as $B(n, p)$

Then

$P(x = r) \ = nc_x p^x q^{n-x}$

- For x considered as approximate by $N(np, npq)$

Then

$p(x = r) = p[r - 0.5 \leq r \leq r + 0.5]$

$= \emptyset(r + 0.5) - \emptyset(r - 0.5)$

Questions

24: A fair win is tested 400 times; find the probability of obtaining between 190 and 210 heads inclusive.

Solution

Given

N= 400

P = ½

B = (400, ½)

Also

$$ee = np$$

$$= 400\ x\frac{1}{2}$$

$$= 200$$

$$\sigma^2 = np$$

$$= 400\ x\ ½\ x\ ½$$

$$= 100$$

$$\mathrm{N}\ (200{,}100)$$

$$\therefore B\ \left(400,\frac{1}{2}\right) - N(200,100)$$

$$p(190 \le x \le 210) = P(189.5 \le x \le 210.5)$$

From

$$2 = \frac{x-ee}{\sigma}$$

$$z_1 = \frac{189.5-200}{10}$$

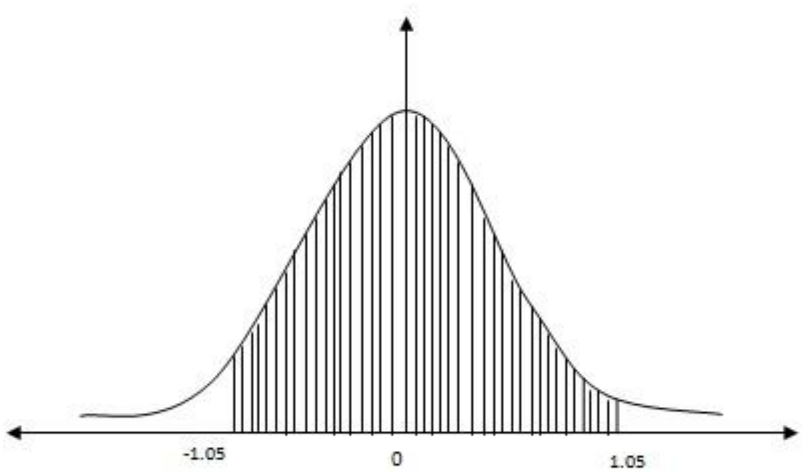
$$z_1 = -1.05$$

$$z_2 = \frac{210.5-200}{10}$$

$$z_2 = 1.05$$

$$p\,(-1.05 \leq z \leq 1.05)$$

Normal curve



$$p(-1.05 \leq z \leq 1.05) = \emptyset(-1.05) + \emptyset(1.05)$$

$$= 2\emptyset(1.05)$$

$$= 0.7063$$

$\therefore$ *The probability is* 0.7063

25. Find the probability of obtaining between 4 and 6 head inclusive in 10 tosses of fair coin.

(a) Using the binomial distribution

(b) Using the normal distribution

Solution

$n = 10$

$r = \frac{1}{2}\ q = \frac{1}{2}$

$B\ 1\left(0, \frac{1}{2}\right)$

$ee = np$

$= 10x\frac{1}{2}$

= 5

$\sigma^2 = npq$

$= 10\ x\frac{1}{2}\ x\frac{1}{2}$

= 2.5

$N(5, 2.5)$

(a) Using the binomial distribution

$p(4 \leq x\ \leq 6)$

$=(x = 4) + p(x = 5) + p(x = 6)$

$= {}^{10}C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6 + {}^{10}C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 + {}^{10}C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4$

${}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$

$\left(\frac{1}{2}\right)^{10}\left({}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6\right)$

= 0.6563

The probability is 0.6563

(b) By using normal distribution

re

$B\left(10, \frac{1}{2}\right) = N(5, 2.5)$

$p(3.5 \leq x \leq 6.5)$

$z = \frac{x - ee}{\sigma}$

$z_1 = \frac{3.5 - 5}{\sqrt{2.5}}$

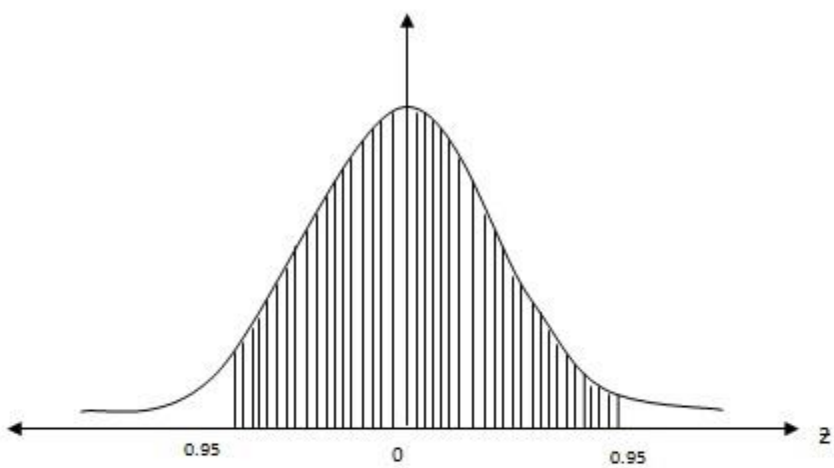
$z_1 = -0.95$

$z_2 = \frac{6.5 - 5}{\sqrt{2.5}}$

$z_2 = 0.95$

$p(-0.05 \leq z \leq 0.95)$

Normal curve



$$p(-0.95 \leq z \leq 0.95) = \emptyset(-0.95) + \emptyset(0.95)$$

$$= 2\emptyset 0.95$$

$$= 0.6579$$

$\therefore$ The probability is 0.6579

(26) Find the probability of obtaining form 40 to 60 heads in 100 tosses of a fair coin

(27) (a) A binomial experiment consists of "n" trials with a probability of success "p" an each trial.

(i) Under what condition be used to approximate this binomial distribution.

(ii) Using the conditions named in (i) above, write down mean $(\bar{x})$ and standard deviation

(b) The probability of obtaining head is ½ when a fair coin is tossed 12 times.

(i) Find the mean $(\bar{x})$ and standard deviation for this experiment

(ii) Hence or otherwise, approximate using normal distribution the probability of getting heads exactly 7 times

Solution #27

(a) (i) The condition are

Solution

The condition are

$n>50$

p is not too large or too small

$(0.2 \leq p \leq 0.8)$

ii) mean $(\bar{x}) = np$

$$Standard\ devaiation\,(\delta) = \sqrt{np(1-p)}$$

$$\sqrt{12 * \frac{1}{2} * \frac{1}{2}}$$

$$\sqrt{3}$$

$= 1.7321$

iii) (x=7)

B(12,1/2) = N(6,3)

P(0.5 $\leq \times \leq 7.5$)

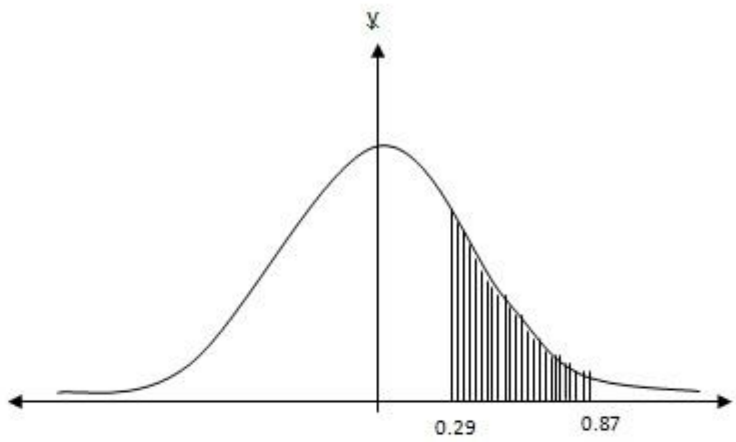
$$Z_1 = \frac{x-\mu}{\delta}$$

$$Z_1 = \frac{6.5 - 6}{\sqrt{3}}$$

$Z_1 = 0.29$

$$Z_2 = \frac{7.5 - 6}{\sqrt{3}}$$

$Z_2 = 0.87$



P(0.29 ≤ Z ≤ 0.87) = $\Phi(0.87) - \Phi(0.29)$

The probability of getting head exactly of 7 times is 0.1938.

28) A machine producing rulers of normal length 30cm is examined carefully and found to produce rulers whose actual lengths are distributed as N(30,0.0001) Find the probability that a ruler chosen at random has a length between 30cm and 30.01 cm

Soln # 28

N (30,0.0001)

μ=30, $\delta = 0.01$

P (30 ≤ × ≤ 30.01)

Z = $\frac{x-\mu}{\delta}$

Z_1= 30-30

0.01

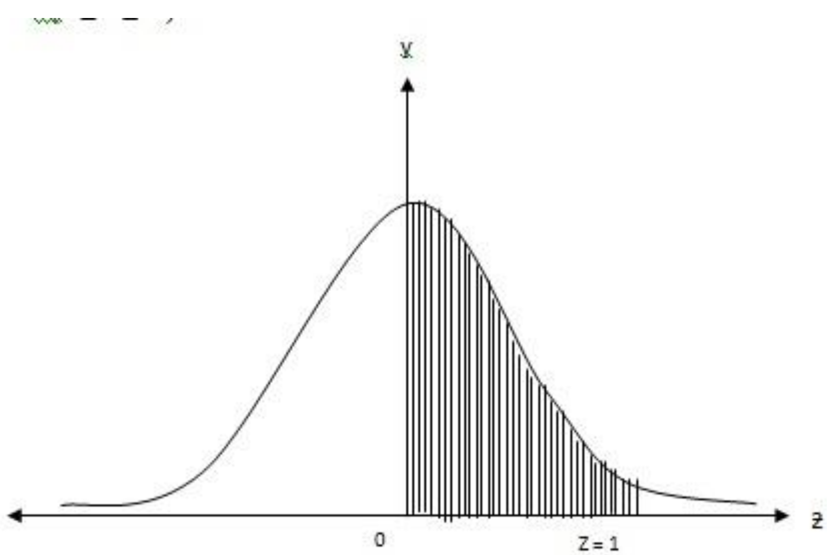
$Z_1 = 0$

$Z_2 = 30.01-30$

$=0.01$

$Z_2 = 1$

$P(0 \le z \le 1)$



P (0 ≤ z ≤ 1) = É¸ (1)

= 0.3413

Probability is 0.3413

COMPLEX NUMBER

The solution of a quadratic equation $ax^2 + bx + c$ can be obtained by the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

i) If $2x^2 + 9x + 7 = 0 \rightarrow x = \frac{-9 \pm \sqrt{81 - 56}}{4}$

$$= \frac{-9 \pm \sqrt{25}}{4}$$

$$= \frac{-9 \pm 5}{4}$$

$$\therefore x = 1 \; or \; x = \frac{-7}{2}$$

This is straight forward enough.

ii) If $5x^2 - 6x + 5 = 0 \; in \; the \; same \; way \; we \; get x = \frac{6 \pm \sqrt{36 - 100}}{10}$

$$= \frac{6 \pm \sqrt{-64}}{10}$$

In fact $\sqrt{-64}$ cannot be represented by an ordinary number.

$\rightarrow -64 = -1 \times 64$
$ie\ \sqrt{-64} = \sqrt{-1} \times \sqrt{64} = 8\sqrt{-1}$
$\sqrt{-64} = 8\sqrt{-1}$
$let\ i = \sqrt{-1}$
$then\ 8\sqrt{-1} = 8.i$
$= 8i$

Similarly,

$\sqrt{-36} = \sqrt{-1}.\sqrt{36} = 6i$
$\sqrt{-49} = \sqrt{-1}.\sqrt{49} = 7i$

Then from the equation that we have been solving it gives that

$$x = \frac{6 \pm \sqrt{-64}}{10}$$
$$x = \frac{6 \pm 8i}{10}$$
$$\therefore x = 0.6 + 0.8i\ or\ 0.6 - 0.8i$$

Since i stand for $\sqrt{-1} \rightarrow i = \sqrt{(-1)}$
$\therefore i^2 = -1$

Note that $\sqrt{-1}$

$i^2 = -1$
$i^3 = (i^2)i = -1.i = -i$
$i^4 = (i^2)^2 = (-1)^2 = 1$
$i^5 = (i^2)^2.i = (-1)^2.i = i$
$i^6 = (i^2)^2.(i^2) = (-1^2).(-1) = 1$

The power of i reduces to one of

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

It can be deduced that

Numbers of the form $a + ib$,where a and b are real numbers are called complex numbers.

Note that $a + ib$ cannot be combined any further

In such expression

A is called the real part of a complex number

B is called imaginary part of a complex number (NOT ib)

Complex number = (real part) + (imaginary part)

OPERATION ON COMPLEX NUMBERS

ADDITION AND SUBTRACTION

Examples:

1. $(4 + 5i) + (3 - 2i)$

Solution

$(4+5i)+(3-2i)=(4+3)+(5i-2i)$

$=7+3i$

2. $(4+7i)-(2-5i)$

Solution

$(4+7i)-(2-5i)=(4-2)+(7i+5i)$

$=2+12i$

So in general $[(a+ib)+(c+id)=(a+c)+i(b+d)]$

EXERCISE

I. $(6+5i)-(4-3i)+(2-7i)$

II. $(3+5i)-(5-4i)-(-2-3i)$

MULTIPLICATION OF COMPLEX NUMBERS

Example:

1. $(3+4i)(2+5i)$

Solution

$(3+4i)(2+5i)=6+15i+8i+20i^2$

$=6+23i-20$

$=-14+23i$

2. $(3+4i)(2-5i)(1-2i)$

Solution

$(3+4i)(2-5i)(1-2i) = [(3+4i)(2-5i)](1-2i)$

$= [6-15i+8i-20i^2](1-2i)$

$= (26-7i)(1-2i)$

$= 26-52i-7i+14i^2$

$= 12-59i$

3. $(2+3i)(2-3i) = 4-6i+6i-9i^2$

$= 4+9$

= 13

Any pair of complex numbers of the form $a \pm ib$ has a product which is real.

i.e.

$(a+bi)(a-bi) = a^2 - abi + abi - bi^2$

$= a^2 + b^2$

Such complex numbers are said to be *conjugate*

Each is a conjugate of the other.

Hence

$If\ a+ib\ \ is\ denoted\ by\ Z\ then\ the\ conjugate\ a-ib\ is\ denoted\ by\ \bar{Z}$

i.e.

.if $Z = a + ib$ then the conjugate is $\bar{Z} = a - ib$

-

A division will be done by *multiplying numerator* and *denominator* the *conjugate* of the denominator.

Example

$find\ \frac{2+9i}{5-2i}$

For division, the numerator and denominator both will be multiplied by the conjugate of the denominator.

i.e.

$$\frac{2+9i}{5-2i} = \frac{(2+9i)(5+2i)}{(5-2i)(5+2i)}$$
$$= \frac{10+4i+45i+18i^2}{25+10i-10i-4i^2}$$
$$= \frac{-8+49i}{29}$$
$$= \frac{-8}{29} + \frac{49i}{29}$$

NOTE: - The complex number is zero if and only if the real term and the imaginary term are each zero.

- The real term is given first even when is negative

i.e.

$X + Yi = 0\ if\ and\ only\ if\ X = 0 = Y$

Suppose

$a + bi = c + di$

$\rightarrow a + bi - (c + di) = 0$

$(a - c) + (b - d)i = 0$

$a = c \; and \; b = d$

Thus *two complex numbers are equal if and only if the real terms and the imaginary terms are separately equal.*

Example:

Find the value of x and y if

a) $(x + yi) = (3 + i)(2 - 3i)$

Solution

$(x + yi) = (3 + i)(2 - 3i)$

$6 - 9i + 2i - 3i^2$

$= 6 - 7i + 3$

$= 9 - 7i$

$\therefore x = 9 \; and \; yi = -7i$

$y = -7$

$x = 9 \; and \; y = -7$

b) $\frac{2+5i}{1-i} = x + yi$

Solution

$$\frac{2+5i}{1-i} = x + yi$$

$$2 + 5i = (1 - i)(x + yi)$$

$$= x + yi - xi - yi^2$$

$$5 + 5i = x + y + (y - x)i$$

$$\rightarrow \begin{cases} x + y = 2 \ldots\ldots\ldots (i) \\ -x + y = 5 \ldots\ldots\ldots ((ii) \end{cases}$$

$$2y = 7$$

$$y = \frac{7}{2}$$

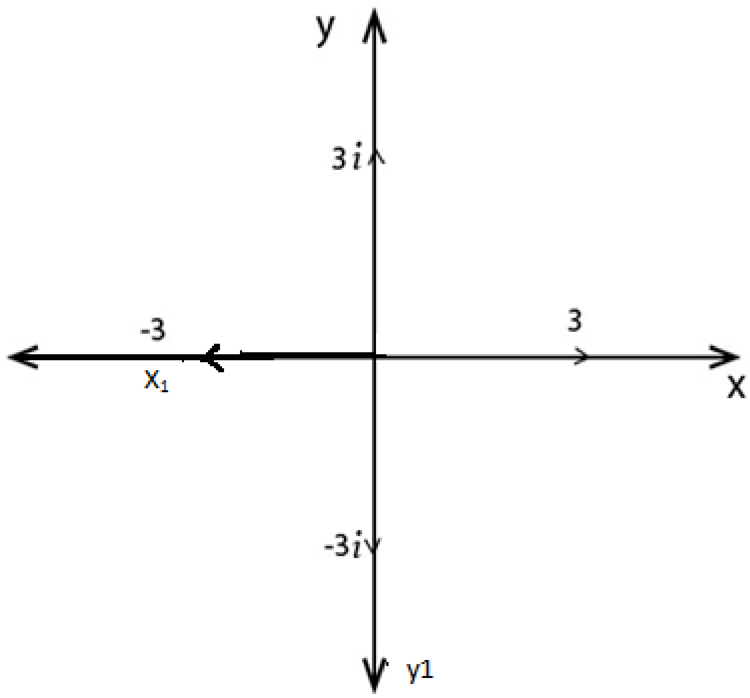
$$x = 2 - \frac{7}{2}$$

$$= \frac{-3}{2}$$

$$\therefore x = \frac{-3}{2} \ and \ y = \frac{7}{2}$$

GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS

Consider the reference line denoted by XX_1 and YY_1

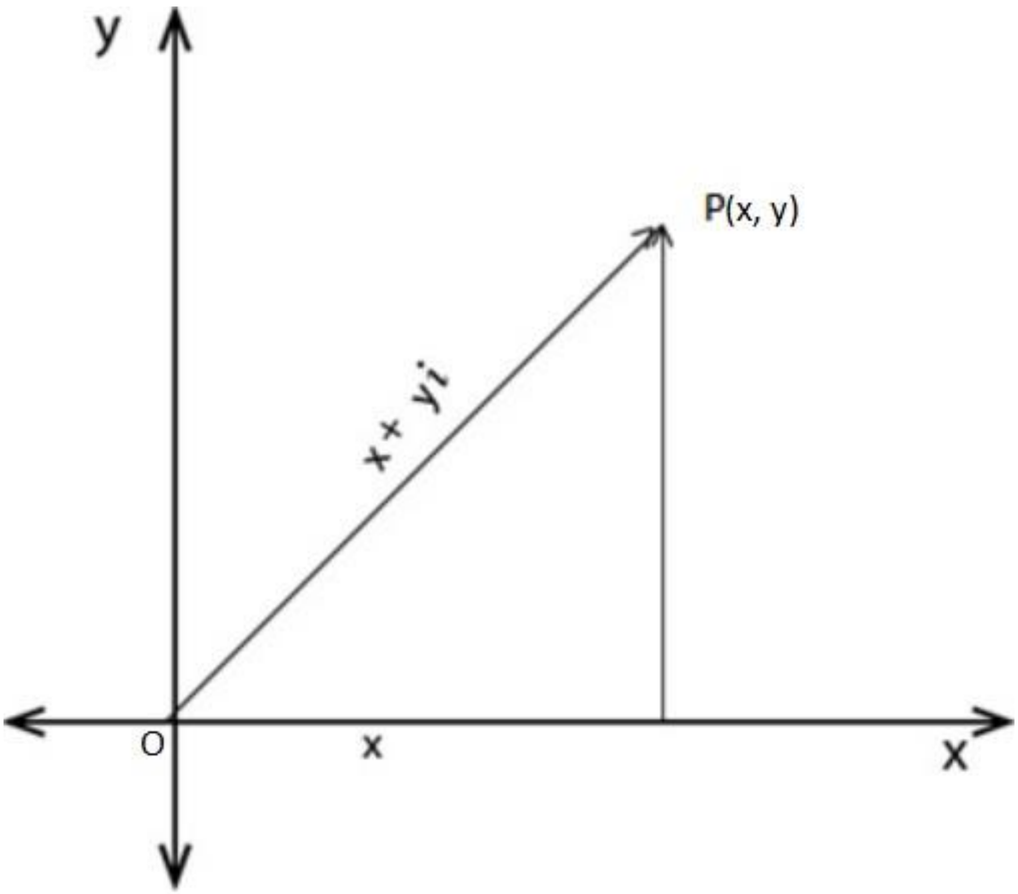


i) x- axis represents real number (i.e. XX_1 is called real axis

ii) y- axis represents imaginary number (YY_1 is called imaginary axis

ARGAND DIAGRAM

If $x + yi$ is a complex number this can be represented by the line $\overline{OP}$ where P is the point (x, y)



This graphical representation constitutes an Argand diagram

Example:

Draw an Argand diagram to represent the vectors

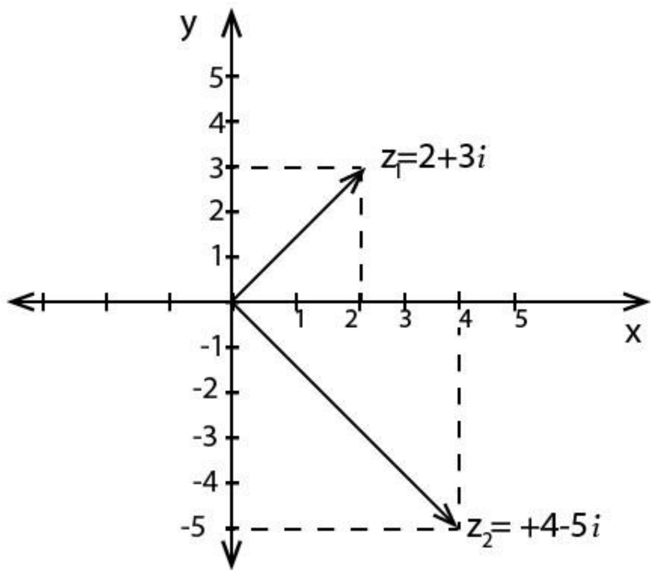
i) $Z = 2 + 3i$

ii) $-4 - 5i = Z_2$

Z is often used to denote a complex

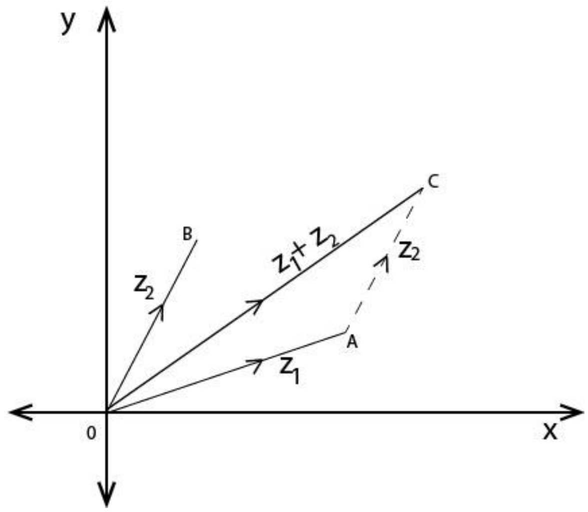
Solution

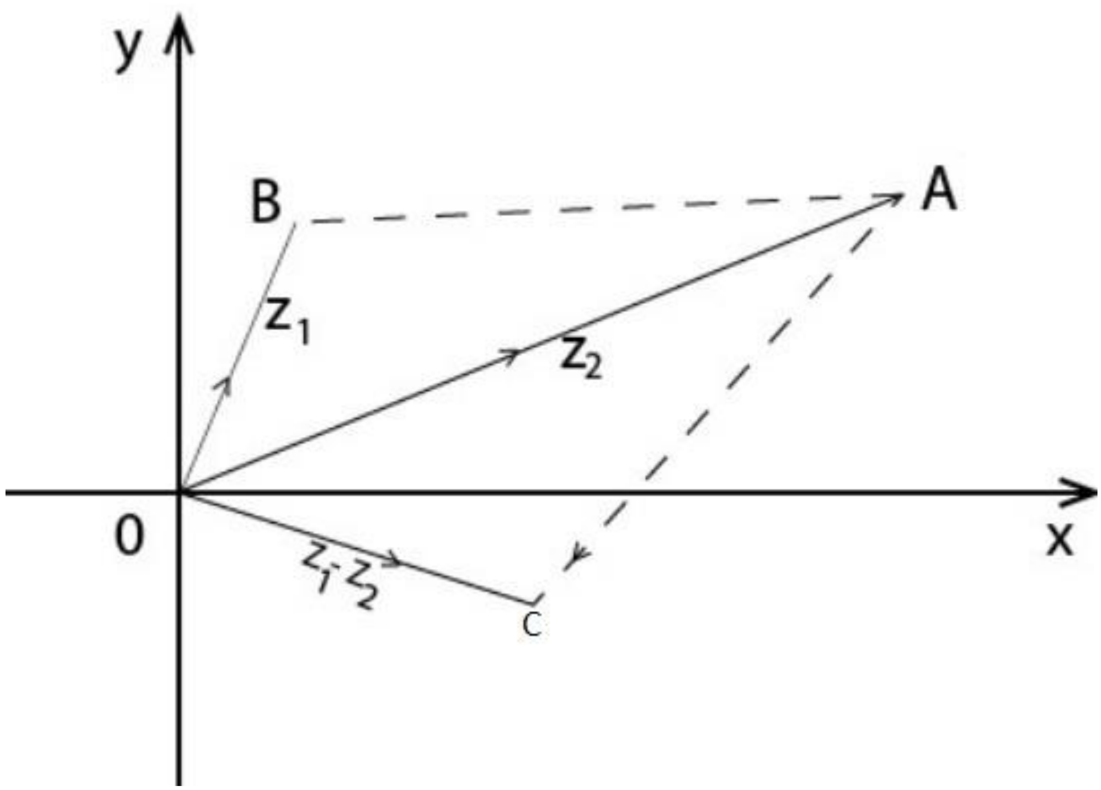
Using the same XY – plane



GRAPHICAL ADDITION AND SUBTRACTION

Consider Z and Z_2 representing an Argand diagram





Taking $Z_1 = x_1 + y_1 i$ and $Z_2 = x_2 + y_2 i$ .The coordinates of C are

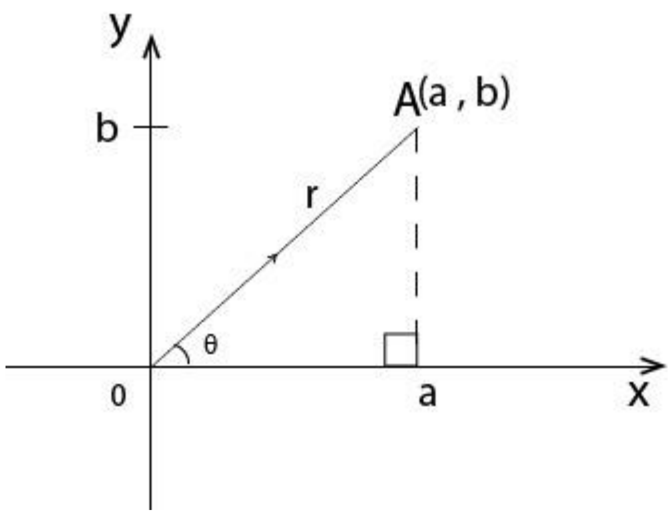
$[(x_1 + x_2), (y_1 + y_2)]$

Hence $\overline{OC}$ represented the complex number

$[(x_1 + x_2), (y_1 + y_2)]$.

MODULUS AND ARGUMENT

Let $a + ib$ be the complex number which suggests that $\overrightarrow{OA}$ represents $a + ib$ and A (a, b) is the point



r is the length of OA

θ Is the angle between the positive x axis and $\overrightarrow{OA}$

$$r^2 = a^2 + b^2$$

$$r = \sqrt{a^2 + b^2}$$

OA is called the modulus of complex number $a + ib$

i.e.

$[|a+bi| = r = \sqrt{a^2+b^2}]$

The angle θ is called the argument of $a+ib$ and written arg($a+ib$)

$\left[\arg(a+bi) = \theta = \arctan\left(\frac{b}{a}\right)\right]$

Note:

The position of OA is unique and corresponds to only one value of θ in the range $-\pi < \theta < \pi$

An argument is also known as amplitude

- To find the argument of $a+bi$ we use $\arctan\left(\frac{b}{a}\right)$ together with quadrant diagram

Example

Find the argument of each of the following complex numbers

a) $4+3i$

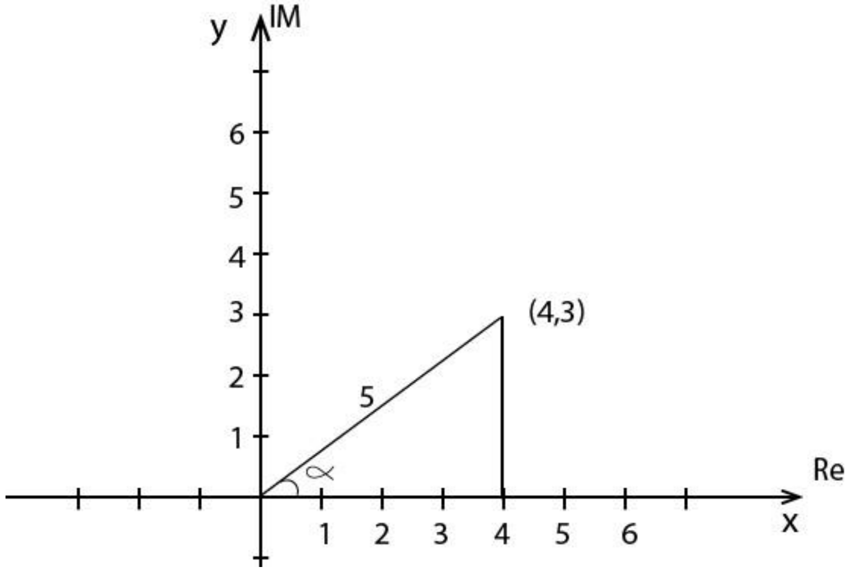
b) $-4+3i$

c) $-4-3i$

d) $4-3i$

Solution

(a) 4+3i



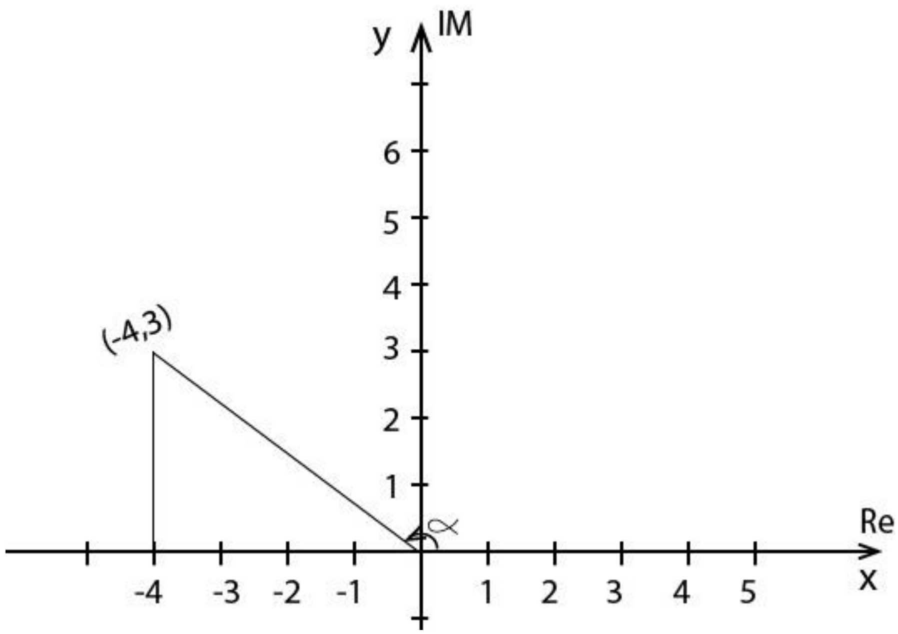
$r = \sqrt{4^2 + 3^2} = 5$

$\arg(4 + 3i) = \arctan\left(\frac{3}{4}\right) = 0.644^c$

$\propto = 0.644°$

Solution

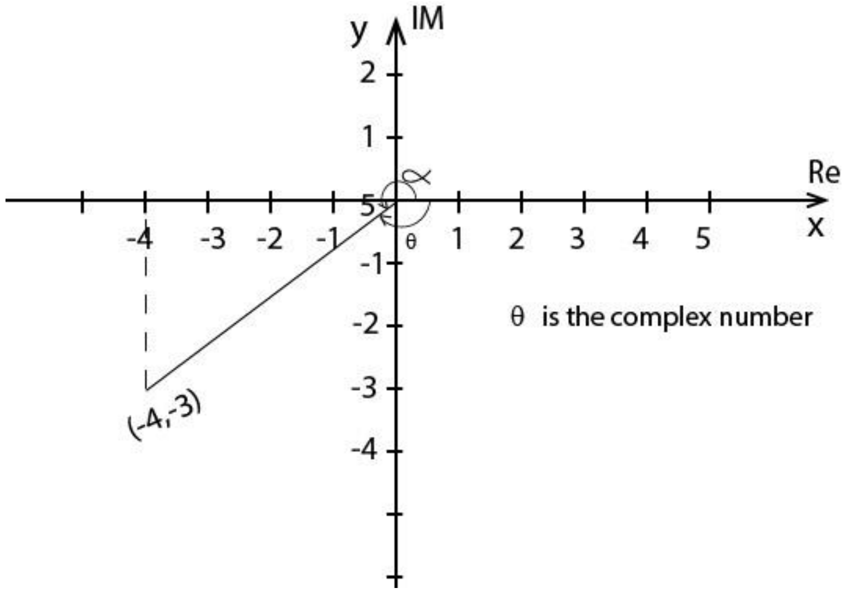
(b) $-4 + 3i$



$\alpha = \arctan\left(\frac{-3}{4}\right) = 2.498^c$

Solution

(c)-4-3i

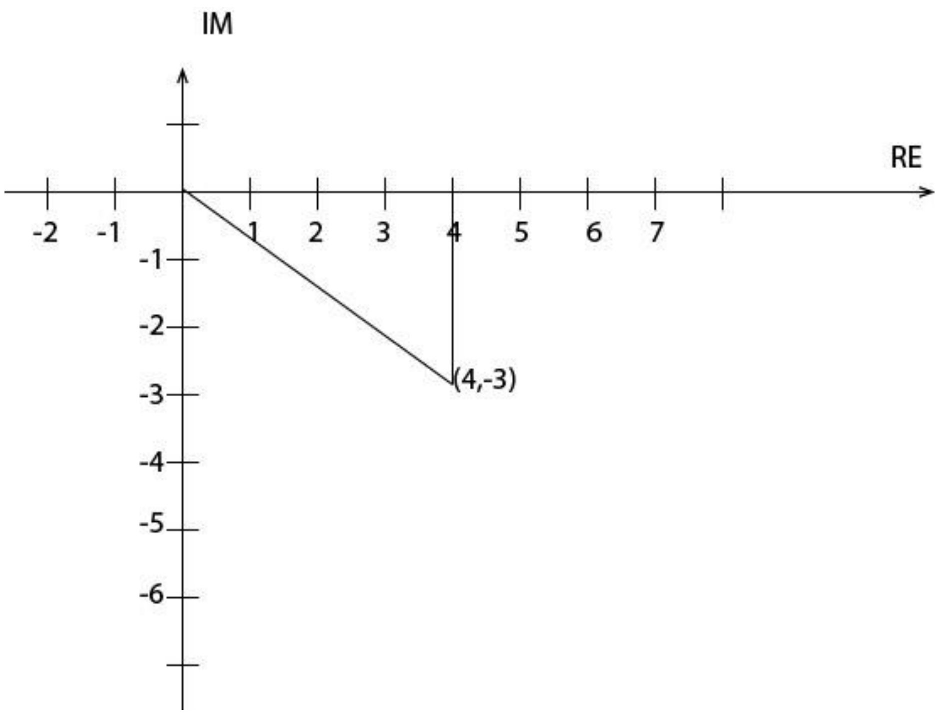


$\alpha = \arctan\left(\frac{-3}{-4}\right) = 3.785^c$

$-180 < \arg(x+yi) \leq 180. \arg(x+yi)$ *is the principle value*

Solution

(d) $4-3i$



$\alpha = \arg(4-3i) = \arctan\left(\frac{-3}{4}\right) = -0.644^c$

EXERCISE

Represent the following complex numbers by lines on Argand diagrams. Determine the modulus and argument of each complex number

a) $3-2i$

b) $(3+i)(4+i)$

SQUARE ROOTS OF COMPLEX NUMBERS

Example

Find $\sqrt{15+8i}$

Solution

$\sqrt{15+8i} = a+bi$
$15+8i = (a+bi)^2$
$= a^2 + 2abi + b^2i^2$

$a^2 + 2abi - b^2$
$= a^2 - b^2 - 2abi$
$\rightarrow 15 = a^2 - b^2 \; and \;\; 8i = 2abi$
$8 = 2ab$

$4 = ab$
$a = \frac{4}{b}$

$putting \; a = \frac{4}{b} \; into \; 15 = a^2 - b^2$

$\rightarrow 15 = \left(\frac{4}{b}\right)^2 - b^2$
$15 = \frac{16}{b^2} - b^2$

$15b^2 = 16 - b^4$

$b^4 + 15b^2 - 16 = 0$

$b^4 - b^2 + 16b^2 - 16 = 0$

$b^2(b^2-1) + 16(b^2-1) = 0$

$(b^2+16)(b^2-1)=0$

$either\ b^2+16=0\ or\ b^2=1$

$\therefore b^2=1$

$b=\pm 1$

$using\ a=\frac{4}{b}$

$\rightarrow \sqrt{15+8i}=\pm(4+i)$

Example;

Given that $2+3i$ is a root of the equation $Z^3-6Z^2+21Z-26=0$ find the other two roots

Solution

Polynomial has real coefficient $2+3i$ and its conjugate is a root of the polynomial

$\rightarrow Z-(2+3i) and\ Z-(2-3i) are\ factors\ then\ \left(Z-(2+3i)\right)\left(Z-(2-3i)\right)\ is\ a\ factor$

$\rightarrow \left(Z-(2+3i)\right)\left(Z-(2-3i)\right)$

$=Z^2-Z(2-3i)-Z(2+3i)+(2+3i)(2-3i)$

$=Z^2-4Z+4-9i^2$

$=Z^2-4Z+4+9$

$= Z^2 - 4Z + 13 \text{ is a factor}$

⇒To find the other factor of $z^3 - 6z^2 + 21z - 26 = 0$

$$
\begin{array}{r|l}
 & \qquad\qquad\qquad\quad Z-2 \\
\hline
 & Z^3 - 6Z^2 + 21Z - 26 \\
 & -Z^3 - 4Z^2 + 13Z \\
Z^2 - 4Z + 13 & \qquad -2Z^2 + 8Z - 26 \\
 & \quad -(-2Z^2 + 8Z - 26) \\
 & \qquad - \quad - \quad -
\end{array}
$$

$\therefore Z - 2 \text{ is a factor}$

$\rightarrow \text{Roots are } 2, 2 + 3i \text{ and } 2 - 3i$

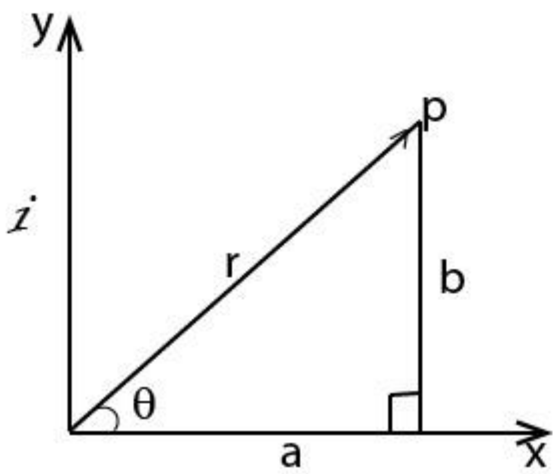
EXERCISE

1. Solve the following equation $2x^2 + 7x + 1 = 0$

2. Given that $Z = x + yi$ express the complex number $\frac{Z+i}{Zi+2}$ in polynomial form hence find resulting complex when $Z = 1 + 2i$

POLAR FORM OF A COMPLEX NUMBER

If $Z = a + ib$ then Z can be written in polar form i.e. in terms r and θ



Let OP be a vector $a + ib$
r be the length of the vector

θ be the angle made with OX

From the diagram (Argand diagram) we can see that

$$\cos\theta = \frac{a}{r} \to a = r\cos\theta$$
$$\sin\theta = \frac{b}{r} \to b = r\sin\theta$$
this gives

$$Z = r\cos\theta + ri\sin\theta$$
$$ieq\ Z = r(\cos\theta + i\sin\theta)$$

$$where\ r = |Z| = \sqrt{a^2 + b^2}\ and\ \theta = \tan^{-1}\frac{b}{a}$$

Example

Express $4 + 3i$ in polar form

Solution

$r = |Z| = \sqrt{4^2 + 3^2}$
$= 5$

$\theta = \arg(Z) = \arctan\left(\frac{3}{4}\right)$
$= \backslash 0.644^c$

$\therefore 4 + 3i = 5(\cos(0.644^c) + i\sin(0.644^c))$

NOTE

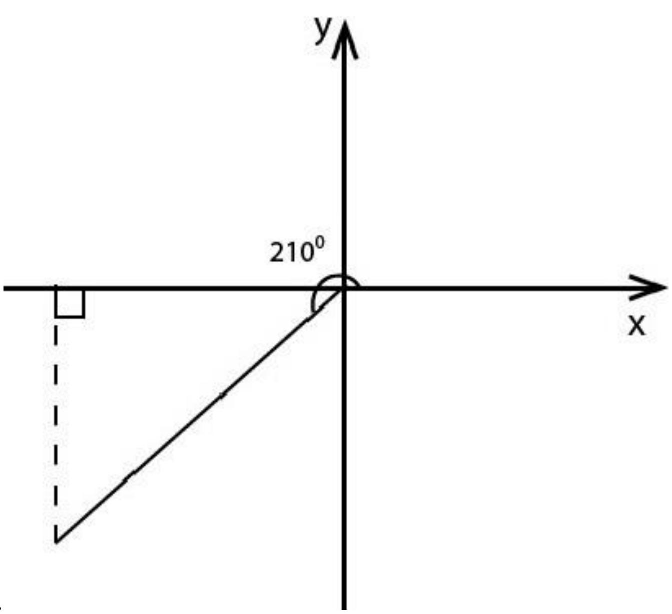
If the argument is greater than 90, care must be taken in evaluating the cosine and sine to include the appropriate signs.

E.g.

Express in the form $a + ib, if\ Z = 2(\cos 210° + i\sin 210°)$

Solution

Since the vector lies in the 3rd quadrant



i.e.

$\cos 210 = -\cos 30$â—¦

$\sin 210 = -\sin 30$â—¦

$then\ Z = 2(-\cos 30° - i \sin 30°)$

$= -1.732 - i$

CONJUGATES IN POLAR FORM

$If\ Z = r(\cos\theta + i \sin\theta) \rightarrow\ Z^* = r(\cos\theta - i \sin\theta)$

$where\ \cos\theta = \cos(-\theta), -i \sin\theta = i \sin(-\theta)$

$Z^* = r(\cos(-\theta) + i \sin(-\theta))$

NB:

Taking the conjugate in polar form changes the sign of its argument

Example

Express $Z = 3 + 4i$ in polar form and then find its conjugate

Solution

$r = \sqrt{3^2 + 4^2}$

$= 5$

$\theta = \arg(3 + 4i) = \arctan\left(\frac{4}{3}\right)$

$\theta = 0.927^c$

$\therefore Z = 5[\cos(0.927^c) + i\sin(0.927^c)]$

$Z^* = 5[\cos(-0.927^c) + i\sin(-0.927^c)]$

$let\ Z_1 = r_1(\cos\theta + i\sin\theta), Z_2 = r_2(\cos\beta + i\sin\beta)$

$then$

$Z_1Z_2 = r_1(\cos\theta + i\sin\theta)r_2(\cos\beta + i\sin\beta)$

$= r_1r_2(\cos\theta\cos\beta + i\cos\theta\sin\beta + i\sin\theta\cos\beta + i^2\sin\theta\sin\beta)$

$= r_1r_2(\cos\theta\cos\beta - \sin\theta\sin\beta + i(\cos\theta\sin\beta + \sin\theta\cos\beta))$

$= r_1r_2[\cos(\theta + \beta) + i\sin(\theta + \beta)]$

$\therefore Z_1Z_2 = r_1r_2[\cos(\theta + \beta) + i\sin(\theta + \beta)]$

$similarly$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\frac{(\cos\theta + i\sin\theta)}{(\cos\beta + i\sin\beta)} \times \frac{(\cos\beta - i\sin\beta)}{(\cos\beta - i\sin\beta)}$$

$$= \frac{r_1}{r_2}\left[\frac{(\cos\theta\cos\beta - i\cos\theta\sin\beta + i\sin\theta\cos\beta - i^2\sin\theta\sin\beta)}{cos^2\beta - i^2sin^2\beta}\right]$$

$$= \frac{r_1}{r_2}(\cos\theta\cos\beta + \sin\theta\sin\beta + i(\sin\theta\cos\beta - \cos\theta\sin\beta))$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}(\cos(\theta - \beta) + i\sin(\theta - \beta))$$

Example

If $Z_1 = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), Z_2 = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Find i) Z_1Z_2 ii) $\frac{Z_1}{Z_2}$

Solution

i) $Z_1Z_2 = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \times 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 12\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right]$

$\therefore\ Z_1Z_2 = 12\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{6}\right]$

ii) $\frac{Z_1}{Z_2} = \frac{3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}$

$= \frac{3}{4}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$

$= \frac{3}{4}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]$

DEMOIVRE'S THEOREM

Demoivre's theorem is a generalized formula to compute powers of a complex number in its polar form

Consider $Z = r(\cos\theta + i\sin\theta)$ from the earlier discussion we can find (Z)(Z)

$\rightarrow (Z)(Z) = Z^2 = r(\cos\theta + i\sin\theta) \times r(\cos\theta + i\sin\theta)$

$= r^2(cos^2\theta + i\cos\theta\sin\theta + i\sin\theta\cos\theta + i^2sin^2\theta)$

$= r^2\left(cos^2\theta - sin^2\theta + i(\cos\theta\sin\theta + \sin\theta\cos\theta)\right)$

$= r^2(\cos 2\theta + i\sin 2\theta)$

$\therefore Z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$\rightarrow Z^3 = Z^2 Z = r^2(\cos 2\theta + i\sin 2\theta) \times r(\cos\theta + i\sin\theta)$

$= r^3[\cos 2\theta\cos\theta + i\cos 2\theta\sin\theta + i\sin 2\theta\cos\theta + i^2\sin 2\theta\sin\theta]$

$= r^3[\cos 2\theta\cos\theta - \sin 2\theta\sin\theta + i(\cos 2\theta\sin\theta + \sin 2\theta\cos\theta)]$

$= r^3[\cos(2\theta + \theta) + i\sin(2\theta + \theta)]$

$= r^3(\cos 3\theta + i\sin 3\theta)$

This brings us to Demoivres theorem

If $Z = r(\cos\theta + i\sin\theta)$ and n is a positive integer

Then

$Z^n = r^n(\cos n\theta + i\sin n\theta)$

Proof demoivre's theorem by induction.

Test formula to be true for n

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

For n = 1

$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

L.H.S = R.H.S

For n = 2

$(\cos\theta + i\sin\theta)^2 = cos^2\theta + 2i\cos\theta\sin\theta - sin^2\theta$

$= cos^2\theta - sin^2\theta + 2i\cos\theta\sin\theta$

$= cos^2\theta - sin^2\theta + 2i\sin\theta\cos\theta$

$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

L.H.S = R.H.S

For n = K

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Let us show that the formula is true for n = k+1

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$

Consider R.H.S

$= (\cos\theta + i\sin\theta)^k \ \cos\theta + i\sin\theta$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\cos\theta\sin k\theta - \sin k\theta\cos\theta$

$= \cos k\theta\cos\theta - \sin k\theta\cos\theta + i(\cos k\theta\sin\theta + \cos\theta\sin k\theta)$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

$\therefore (\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

Since the formula was shown to be true for n = 1, 2 hence its true for integral value of n.

Example

1. Find $\left(1+\sqrt{3i}\right)^2$

Solution

$let\ Z = 1 + \sqrt{3i}$

$\rightarrow |Z| = \sqrt{1^2 + (\sqrt{3})^2}$

$= \sqrt{4}$

$= 2$

$\theta = \arctan\left(\frac{\sqrt{3}}{1}\right)$

$\theta = \frac{\pi}{3}$

by demoivre'stheorem

$Z^n = r^n(\cos n\theta + i \sin n\theta)$

$\rightarrow Z^5 = \left[R\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)\right]^5$

$= 2^5\left(\cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}\right)$

$\therefore (1 + \sqrt{3i})^5 = 32\left(\cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}\right)$

2. From Demoivere's theorem prove that the complex number $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n$ is always real and hence find the value of the expression when n = 6

Solution

$let\ Z_1 = \left(\sqrt{3} + i\right)^n\ and\ Z_2 = \left(\sqrt{3} - i\right)^n by\ demoivre's\ theorem$

$Z_1{}^n + Z_2{}^n = r_1{}^n(\cos n\theta_1 + i\sin n\theta_1) + r_2(\cos n\theta_2 + i\sin n\theta_2)$

$\rightarrow |Z_1| = \sqrt{3+1} = 2\ \ and\ \theta_1 = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$Z_2 = \sqrt{3+1} = 2\quad and\ \ \theta_2 = \arctan\left(\frac{-1}{\sqrt{3}}\right) = \frac{11\pi}{6}$

$$\rightarrow (\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^n\left(\cos n\frac{\pi}{6} + i\sin n\frac{\pi}{6}\right) + 2^n\left(\cos n\frac{11\pi}{6} + i\sin n\frac{11\pi}{6}\right)$$

$$2^n\left[\left(\cos n\frac{\pi}{6} + \cos n\frac{11\pi}{6}\right) + i\left(\sin n\frac{\pi}{6} + \sin n\frac{11\pi}{6}\right)\right]$$

$it\ is\ true\ that\ \cos\beta = \cos(2\pi - \beta)\ \ and - \sin\beta = \sin(2\pi - \beta)$

$$\rightarrow (\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^n\left[\cos n\frac{\pi}{6} + \cos\left(2\pi n - \frac{11\pi n}{6}\right) + i\left(\sin n\frac{\pi}{6} + \sin\frac{2\pi n}{6} - \frac{11\pi n}{6}\right)\right]$$

$$= 2^n\left[\left(\cos n\frac{\pi}{6} + \cos n\frac{\pi}{6}\right) + i\left(\sin n\frac{\pi}{6} - \sin n\frac{\pi}{6}\right)\right]$$

$$= 2^n\left[2\cos n\frac{\pi}{6} + i(0)\right]$$

$$\rightarrow \left(\sqrt{3}+i\right)^n + \left(\sqrt{3}-i\right)^n = 2^n\left(2\cos n\frac{\pi}{6}\right)$$

$when\ n = 6$

$$\left(\sqrt{3}+i\right)^6 + \left(\sqrt{3}-i\right)^6 = 2^6\left(2\cos\frac{6\pi}{6}\right)$$

$2^7 \cos\pi$

$= -128$

FINDING THE n^{th} ROOT

Demoivere's formula is very useful in finding roots of complex numbers.

If n is any positive integer and Z is any complex number we define an n^{th} root of Z to be any complex number 'w' which satisfy the equation $w^n = Z$

$w = \sqrt[n]{Z}$

$ie\ w = Z^{\frac{1}{n}}$

$ie\ w\ is\ the\ n^{th}\ root\ of\ Z\ if\ Z \neq 0$

$Suppose\ w = \rho(\cos\alpha + i\sin\alpha), Z = r(\cos\theta + i\sin\theta)$

$it\ follows\ that$

$\left(\rho(\cos\alpha + i\sin\alpha)\right)^n = r(\cos\theta + i\sin\theta)$

$comparing\ the\ modules\ of\ the\ two\ sides$

$\rho^n = r \to \rho = \sqrt[n]{r}$

$\sqrt[n]{r}\ denotes\ the\ real\ positive\ root\ of\ r$

$more\ over$

$\to \cos n\alpha + i\sin n\alpha = \cos\theta + i\sin\theta$

$\to \cos n\alpha = \cos\theta$

$n\alpha = \theta$

$Hence\ n\alpha = \theta + 2\pi k\ or\ \alpha = \frac{\theta}{n} + \frac{2\pi k}{n}, k = 0, \pm1, \pm2, \pm3, \ldots$

$note\ that\ n\alpha\ and\ \theta\ must\ either\ be\ equal\ or\ differ\ by\ multiple\ of\ 2\pi$

$$\therefore w = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right] \quad k = 0, \pm1, \pm2, \pm3, \ldots$$

$since\ w = Z^{\frac{1}{n}}$

$$hence\ Z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$$

$0, 1, 2, 3, \ldots, n-1$

$since\ other\ values\ of\ K\ produces\ distinct\ values\ w. Hence, there\ are\ exactly\ n\ different\ n^{th}\ roots\ of$

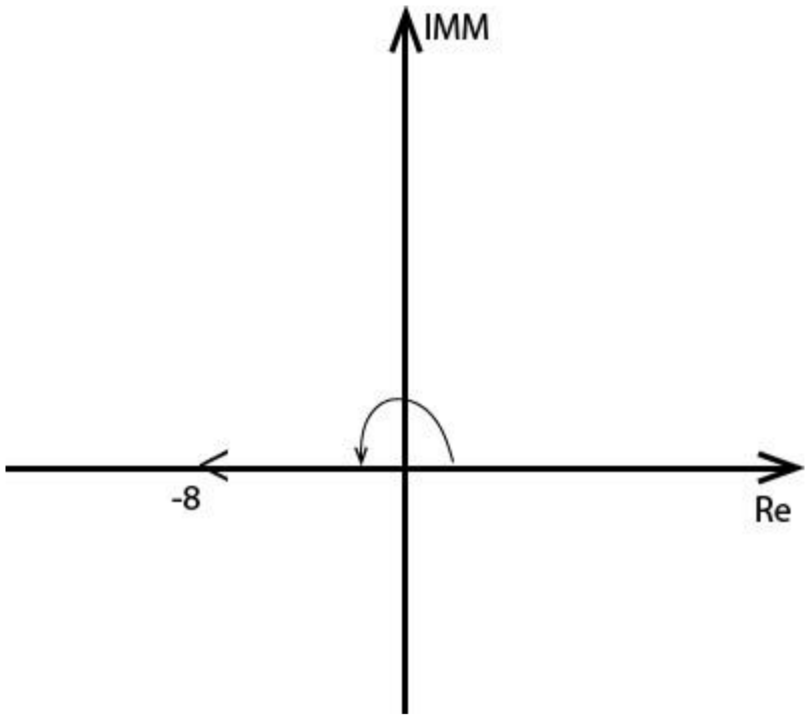
$Z = r(\cos\theta + isin\ \theta)$

Examples

1. Find all cube roots of -8

Solution

-8 lie on the negative real axis



$\rightarrow \theta = \pi.\ since\ \ \cos^{-1}(-1) = \pi\ \ and\ r = \sqrt{(-8)^2 + 0^2} = 8$

$then\ Z = r(\cos\theta + i\sin\theta)$

$\rightarrow -8 = 8(\cos\theta + i\sin\theta)$

$(-8)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\left(\cos\frac{\pi}{3} + \frac{2\pi k}{3}\right) + i\sin\frac{\pi}{3} + \frac{2\pi k}{3}\right]\ \ k = 0,1,2$

$(-8)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\left(\cos\frac{\pi}{3} + \frac{2\pi k}{3}\right) + i\sin\frac{\pi}{3} + \frac{2\pi k}{3}\right]\ \ k = 0,1,2 thus\ cube\ roots\ of -8\ are;$

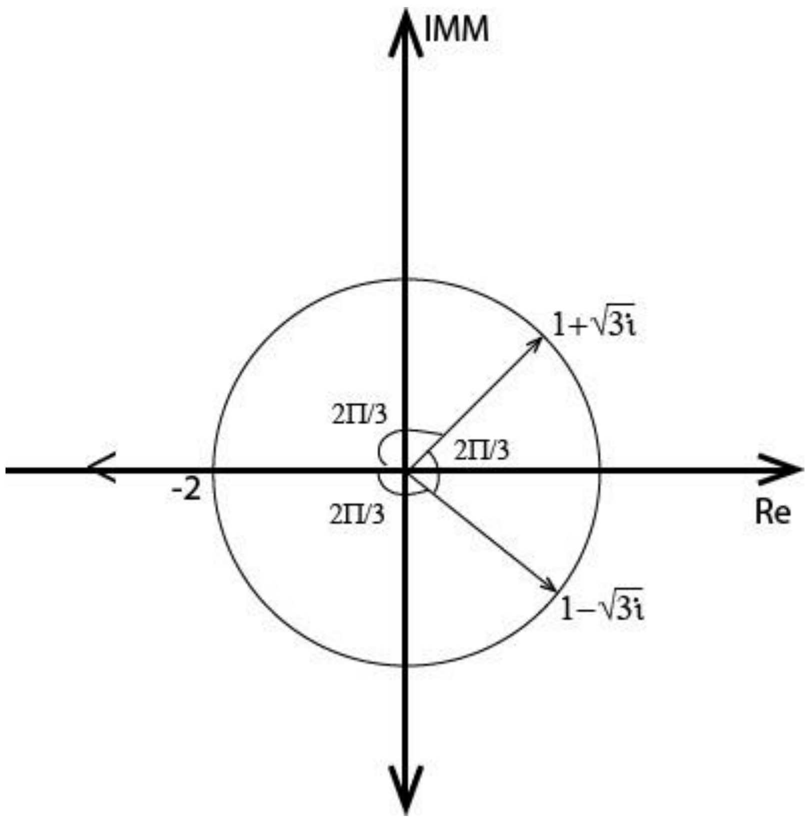
$\rightarrow 2\left(\cos\frac{\pi}{3} + \frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i, for\ k = 0$

$\rightarrow 2(\cos\pi + i\sin\pi) = 2(-1 + 0) = -2\ for\ k = 1$

$$\rightarrow 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i \;\; for\; k = 2$$

$\therefore$ *the roots are* $1 + \sqrt{3}i, -2, 1 - \sqrt{3}i$

these roots can be described as shown in the figure below



2. Solve $Z^3 - 1 = 0$ giving your solution in polar form

Solution:

$Z^3 - 1 = 0$

$Z^3 = 1 + 0i$

$\rightarrow r = \sqrt{1^2 + 0^2} = 1$

$\theta = \arctan\left(\frac{0}{1}\right) = 0°$

$Z^3 = r(\cos\theta + i\sin\theta)$

$Z = r^{\frac{1}{3}}\left[\cos\left(\frac{\theta}{3} + \frac{2\pi k}{3}\right) + i\sin\left(\frac{\theta}{3} + \frac{2\pi k}{3}\right)\right] \; k = 1, 2, 3$

$\rightarrow Z = 1(\cos 0 + i\sin 0) = \cos 0° + i\sin 0° \quad when\ k = 0$

$\rightarrow Z = 1\left[\cos\left(0 + \frac{2\pi}{3}\right) + i\sin\left(0 + \frac{2\pi}{3}\right)\right] = \left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right] \; k = 1$

$\rightarrow Z = 1\left[\cos\left(0 + \frac{4\pi}{3}\right) + i\sin\left(0 + \frac{4\pi}{3}\right)\right] = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} \quad k = 2$

EXERCISE

1. Find all fourth roots of 1.

2. Evaluate $(-1 + i)^{\frac{1}{8}}$

Proving trigonometric identities using Demoivre's theorem

Examples

Prove that

i) $\cos 5\theta \equiv 16cos^5\theta - 20cos^3\theta + 5\cos\theta$

ii) $\sin 5\theta \equiv 16sin^5\theta - 20sin^3\theta + 5\sin\theta$

Solution

Note;

To solve such question you should be aware of the binomial theorem

i) $\cos 5\theta \equiv Re(\cos 5\theta + i\sin 5\theta) \equiv Re(\cos\theta + i\sin\theta)^5$ $by\ demoivre's theorem$

$let\ c = \cos\theta, s = \sin\theta$

$\rightarrow \cos 5\theta = Re(c + is)^5$

$\equiv Re[c^5 + 5c^4 is + 10c^3 s^2 i^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5]$

$\equiv Re[c^5 + 5c^4 is - 10c^3 s^2 - 10c^2 is^3 + 5cs^4 + is^5]$

$\equiv Re[(c^5 - 10c^3 s^2 + 5cs^4) + i(5c^4 s - 10c^2 s^3 + s^5)]$

$\rightarrow \cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$

$cos^5\theta - 10cos^3\theta sin^2\theta + 5\cos\theta\, sin^4\theta$

$But\ sin^2\theta = 1 - cos^2\theta$

$\rightarrow \cos 5\theta \equiv cos^5\theta - 10cos^3\theta[1 - cos^2\theta] + 5\cos\theta\,[1 - cos^2\theta]^2$

$\equiv cos^5\theta - 10cos^3\theta + 10cos^5\theta + 5\cos\theta\,[1 - 2cos^2\theta + cos^4\theta]$

$\equiv cos^5\theta - 10cos^3\theta + 10cos^5\theta + 5\cos\theta - 10cos^3\theta + 5cos^5\theta$

$\equiv 16cos^5\theta - 20cos^3\theta + 5\cos\theta$

$\therefore \cos 5\theta = 16cos^5\theta - 20cos^3\theta + 5\cos\theta$

$in\ a\ similar\ way;$

$i\sin 5\theta \equiv i(5c^4 s - 10c^2 s^3 + s^5)$

$\sin 5\theta \equiv 5cos^4\theta \sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta$

$But\ cos^2\theta = 1 - sin^2\theta$

$then \sin 5\theta \equiv 5(1 - sin^2\theta)^2 \sin\theta - 10(1 - sin^2\theta)sin^3\theta + sin^5\theta$

$\equiv 5(1-2sin^2\theta+sin^4\theta)\sin\theta-10sin^3\theta+10sin^5\theta+sin^5\theta$

$\equiv 5\sin\theta-10sin^3\theta+5sin^5\theta-10sin^3\theta+10sin^5\theta+sin^5\theta$

$\equiv 16sin^5\theta-20sin^3\theta+5\sin\theta$

$\therefore \sin 5\theta=16sin^5\theta-20sin^3\theta+5\sin\theta$

EXERCISE

Show that;

$$\tan 5\theta=\frac{5(tan^4\theta)-2(tan^3\theta)+tan^5\theta}{1-5(2tan^2\theta+tan^4\theta)}$$

$\rightarrow If\ Z=\cos\theta+isin\ \theta, then$

$\frac{1}{Z}=Z^{-1}=(\cos\theta+isin\ \theta)^{-1}=\cos(-\theta)+i\sin(-\theta)\ by\ demoivre's$

$=\cos\theta-i\sin\theta$

$\therefore Z+\frac{1}{Z}=(\cos\theta+isin\ \theta)+(\cos\theta-i\sin\theta)$

$Z+\frac{1}{Z}=2\cos\theta$

$Z-\frac{1}{Z}=(\cos\theta+isin\ \theta)-(\cos\theta-i\sin\theta)$

$=\cos\theta-\cos\theta+i\sin\theta+i\sin\theta$

$=2i\sin\theta$

$\rightarrow Z-\frac{1}{Z}=2i\sin\theta$

These generalised to

$$Z^n + \frac{1}{Z^n} = 2\cos n\theta$$

$$Z^n - \frac{1}{Z^n} = 2i\sin(n\theta)$$

Example

Find an expression for

i) $cos^4\theta$ ii) $sin^3\theta$

Solution

i) We know that $2\cos\theta = Z + \frac{1}{z} \rightarrow \cos\theta = \frac{1}{2}\left(Z + \frac{1}{Z}\right)$

$$= \frac{1}{2}(Z + Z^{-1})$$

$$cos^4\theta = \left(\frac{1}{2}(Z + Z^{-1})\right)^4$$

$$= \frac{1}{16}\left[Z^4 + 4Z^3\left(\frac{1}{Z}\right) + 6Z^2\left(\frac{1}{Z}\right)^2 + 4Z\left(\frac{1}{Z}\right)^3 + \left(\frac{1}{Z}\right)^4\right]$$

$$= \frac{1}{16}\left[Z^4 + \left(4Z^2 + \frac{4}{Z^2}\right) + 6 + \frac{1}{Z^4}\right]$$

$$= \frac{1}{16}\left[\left(Z^4 + \frac{1}{Z^4}\right) + \left(4Z^2 + \frac{4}{Z^2}\right) + 6\right]$$

$$= \frac{1}{16}\left[\left(Z^4 + \frac{1}{Z^4}\right) + 4\left(Z^2 + \frac{1}{Z^2}\right) + 6\right]$$

$$= \frac{1}{16}[2\cos 4\theta + 8\cos 2\theta + 6]$$

$$\therefore cos^4\theta = \frac{1}{8}[\cos 4\theta + 4\cos 2\theta + 3]$$

ii) $2i\sin\theta = Z - \frac{1}{z} \rightarrow \sin\theta = \frac{1}{2i}\left(Z - \frac{1}{Z}\right)$

$\therefore sin^3\theta = \left[\frac{1}{2i}\left(Z - \frac{1}{Z}\right)\right]^3$

$= -\frac{1}{8i}\left[Z^3 - 3Z^2\left(\frac{1}{Z}\right) + 3Z\left(\frac{1}{Z}\right)^2 - \left(\frac{1}{Z}\right)^3\right]$

$= -\frac{1}{8i}\left[\left(Z^3 - \frac{1}{Z^3}\right) + \left(-3Z + 3\frac{1}{Z}\right)\right]$

$= -\frac{1}{8i}\left[\left(Z^3 - \frac{1}{Z^3}\right) + 3\left(-Z + \frac{1}{Z}\right)\right]$

$= -\frac{1}{8i}\left[\left(Z^3 - \frac{1}{Z^3}\right) - 3\left(Z - \frac{1}{Z}\right)\right]$

$= -\frac{1}{8i}[2i\sin 3\theta - 6i\sin\theta]$

$= \frac{1}{4}[-\sin 3\theta + 3\sin\theta]$

EXERCISE

Use Demoivre's theorem to find the following integrals

a) $\int cos^4\theta\, d\theta$

b) $\int(32cos^6\theta - \cos 6\theta)\, d\theta$

c) $\int(\cos 4\theta - 8sin^4\theta)\, d\theta$

THE EULER'S FORMULA (THE EXPONENTIAL OF A COMPLEX NUMBER)

Euler's formula shows a deep relationship between the *trigonometric function* and complex exponential

Since

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \frac{\theta^6}{6!} + \cdots$$

introducing imaginary number i to

$\sin\theta =$

$$ie\ i\sin\theta = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} + \cdots$$

e^{θ}

$$ie\ e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

Re organizing into real and imaginary terms gives.

$$e^{i\theta} = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right]$$

$= \cos\theta + i\sin\theta$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$

Hence if Z is a complex number its exponent form is $Z = re^{i\theta}$ in which

$r = |Z|\ and\ \theta = \arg(Z)$

$Z = re^{i\theta} = r\cos\theta + i\sin\theta$

Example

1. Write $1 + i$ in polar form and then exponential form

Solution

$r = |Z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \arg(1 + i) = \arctan(1) = \frac{\pi}{4}$

$\therefore Z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\ and\ Z = \sqrt{2}e^{i\frac{\pi}{4}}$

2. Express $Z = 3e^{i\frac{\pi}{6}}$ in Cartesian form correct to 2 decimal places

Solution

$r = 3\ and\ \theta = \frac{\pi}{6}$

$\rightarrow Z = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 3(0.8660 + i0.5000)$

$= 2.60 + 1.50i$

$\therefore Z = 2.60 + 1.50i\ \ to\ 2\ d.p$

Note that;

The exponents follow the same laws as real exponents, so that

If $Z_1 = r_1 e^{i\theta_1}$, $Z_2 = r_2 e^{i\theta_2}$ *then*

$$Z_1 Z_2 = r_1 r_2 e^{(i\theta_1 + i\theta_2)} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} e^{(i\theta_1 - i\theta_2)} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

ROOTS

Sometimes you can prefer to find roots of a complex number by using exponential form.

From the general argument $\theta + 2\pi k$

If $Z = r(\cos\theta + i\sin\theta) Z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$

$$hence\, Z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)} \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Example

Find the cube root of Z = 1

Solution

$$Z = 1$$

$$|Z| = 1$$

$$\theta = \arg(1 + 0i) = \arctan(0) = 0°$$

$$\therefore Z_1 = e^{i(0+0)} = 1 \;\; for\; k = 0$$

$\rightarrow Z_2 = e^{i\left(0+\frac{2\pi}{3}\right)}$ $when\ k = 1$ $\rightarrow e^{\frac{2\pi}{3}i}$

$\rightarrow Z_3 = e^{i\left(0+\frac{4\pi}{3}\right)}$ $\rightarrow$ $e^{\frac{4\pi}{3}i}$ $when\ k = 2$

Example 2

Calculate the fifth root of 32 in exponential form

Solution

$Z = 32 + 0i$

$r = \sqrt{32^2 + 0}$

$= 32$

$\theta = \arg(32 + 0i) = \arctan\left(\frac{0}{32}\right) = 0°$

$from\ Z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)}$

$\rightarrow Z_1 = 32^{\frac{1}{5}} e^{i\left(\frac{0}{5}+\frac{2\pi}{5}\times 0\right)} = 2e^{i(0)} = 2 \quad k = 0$

$\rightarrow Z_2 = 2e^{i\left(\frac{2\pi}{5}\right)} = 2e^{\frac{2\pi}{5}i}$ $when\ k = 1$

$\rightarrow Z_3 = 2e^{i\left(\frac{4\pi}{5}\right)} = 2e^{\frac{4\pi}{5}i}$ $when\ k = 2$

$\rightarrow Z_4 = 2e^{i\left(\frac{6\pi}{5}\right)}$ $when\ k = 3$

$\rightarrow Z_5 = 2e^{i\left(\frac{8\pi}{5}\right)} = 2e^{\left(\frac{4\pi}{5}\right)}$ $when\ k = 4$

$\therefore$ $the\ roots\ of\ 32\ are\ 2e^{i(0)}, 2e^{i\frac{2\pi}{5}}, 2e^{i\frac{4\pi}{5}}, 2e^{i\frac{6\pi}{5}}, and\ 2e^{i\frac{8\pi}{5}}$

LOCI OF THE COMPLEX NUMBERS

Complex number can be used to describe lines and curves areas on an Argand diagram.

Example 01

Find the equation in terms of x and y of the locus represented by $|z|=4$

Solution

$let\ Z = x + iy$

$|Z| = |x + iy| = \sqrt{x^2 + y^2}$

$\rightarrow \sqrt{x^2 + y^2} = 4$

$x^2 + y^2 = 4^2$

This is the equation of a circle with centre (0.0) radius 4

Example 02

Describe the locus of a complex variable Z such that $|Z - 2 + 3i| \geq 4$

Solution

$let\ Z = x + iy$

$\rightarrow |Z - 2 + 3i| \geq 4$

$|(x + iy) - 2 + 3i| \geq 4$

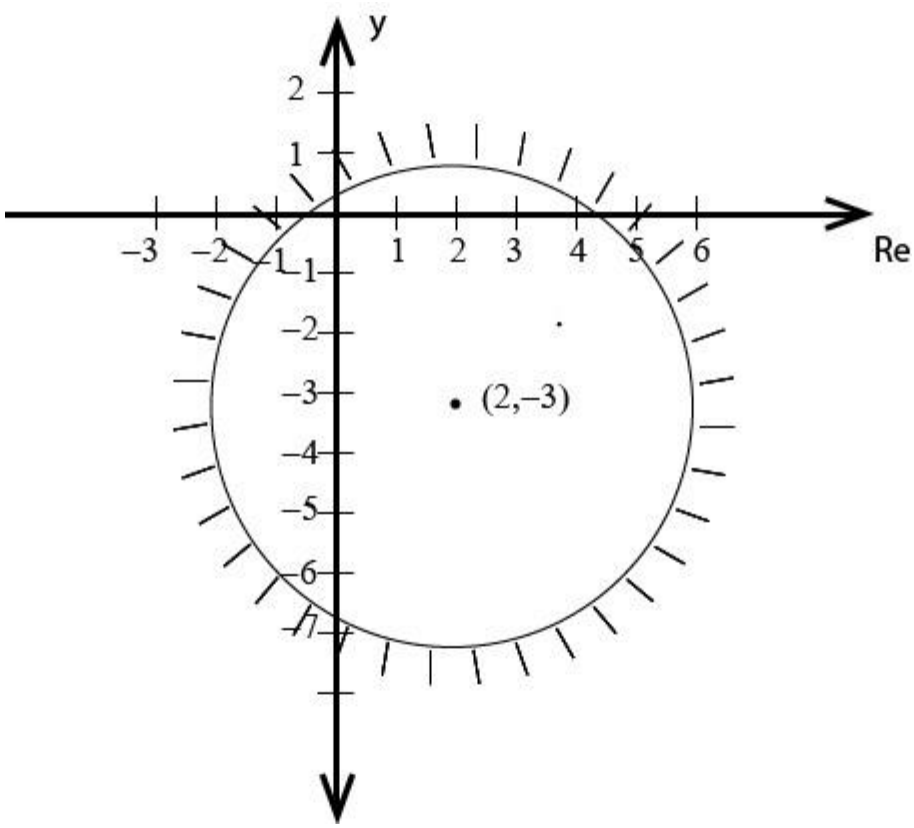
$|x + iy - 2 + 3i| \geq 4$

$|(x - 2) + (y + 3)i| \geq 4$

$\rightarrow \sqrt{(x-2)^2+(y+3)^2} \geq 4$

$(x-2)^2+(y+3)^2 \geq 4^2$

This is the equation of a circle with centre (2,-3), radius 4 in which the point (x, y) lies on and out of the circle.



Example 03

If Z is a complex number, find the locus in Cartesian coordinates represented by the equation $|Z-3|=2$

Solution

$let\ Z = x + iy$

$\rightarrow |Z - 3| = 2\ gives$

$|x + iy - 3| = 2$

$|(x - 3) + iy| = 2$

$\rightarrow (x - 3)^2 + y^2 = 2^2$

$x^2 - 6x + 9 + y^2 = 4$

$x^2 + y^2 - 6x + 9 = 4$

$x^2 + y^2 - 6x + 5 = 0$

This is the needed locus which is a circle with centre (3, 0) and radius 2

Example 04

If Z is a complex number, find the locus of the following inequality

$2 < |Z - (1 + i)| \leq 6$

Solution

We consider in two parts

$2 < |Z - (1 + i)| \leq 6\ \ and\ |Z - (1 + i)| \leq 6$

$let\ Z = x + iy$

$\rightarrow |Z - (1 + i)| = |(x + iy) - (1 + i)|$

$= |x + iy - 1 - i|$

$= |(x-1)+(y-1)i|$

$but\ 2 < |Z-(1+i)|$

$\rightarrow 2 < |(x-1)+(y-1)i|$

$2 < \sqrt{(x-1)^2+(y-1)^2}$

$2^2 < (x-1)^2+(y-1)^2$

$this\ gives\ the\ center\ (1{,}1) and\ radius\ 2$

$take\ |Z-(1+y)| \leq 6$

$|(x-1)+(y-1)i| \leq 6$

$\sqrt{(x-1)^2+(y-1)^2} \leq 6$

$center = (1{,}1), radius = 6$

$\therefore the\ locus\ is\ (x-1)^2+(y-1)^2 \leq 6^2$

$x^2-2x+1+y^2-2y+1 \leq 6^2$

$x^2+y^2-2x-2y+2 \leq 6^2$

$x^2+y^2-2x-2y-34 \leq 0$

DIFFERENTIAL EQUATION

A differential equation is a relationship between the rates of different variables (i.e an independent variables x, a dependent variable, y, and one or more differential coefficient of y with respect to x).

Eg $\frac{x^2 dy}{dx}$ + y sin x = 0

$xy\frac{d^2y}{dx^2} + y\frac{dy}{dx} + e^{3x} = 0$

$\frac{d^2y}{dx2} + \frac{3dy}{dx} - 4y = 0$

<u>The order of a differential equation</u>

The order of a differential equation is given by the highest derivative involved in the equation.

$\frac{xdy}{dx} - y^2 = 0$ is an eqn of the 1st order

$xy\frac{d^2y}{dx^2} - y^2 \sin x = 0$ is an eqn of the 2nd order

$xy\frac{d^3y}{dx^3} - y\frac{d\ \ y}{dx} + e^{4x} = 0$ is an eqn of the 3rd order

<u>The degree of a differential equation</u>

Eg $\frac{x^2dy}{dx} + y \sin x = 0$ the degree is 1b

<u>Linear differential equation (L.D.E)</u>

A linear differential equation should look in the form

$\frac{d^ny}{dx^n} + \frac{p_1d^{n-1}}{dx^{n-1}} + \frac{p_2d^{n-2}y}{dx^{n-2}} + \ldots\ldots\ldots pn - 1\frac{dy}{dx} + pn\,y = Q$

Where P_1, P_2, P_3 …Pn, Q are functions of x, or constant n = 1, 2, 3, 4…….

Examples

i) $\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 6y = 0$

ii) $\frac{d^4y}{dx^4} + \frac{x^2d^3y}{dx} + \frac{x^3dy}{dx} = xe^x$

iii) $\frac{dy}{dx} = 4$

Examples of non L.D.E

$\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 6y = 0$

$\frac{d^2y}{dx^2} + \frac{5y\,dy}{dx} + 6y = 0$

Note

A D. E will be linear if

- Variable y and its derivatives occur at the first degree only
- No product of y and its derivatives
- No transcendental function of y or x

Exercise

Show which of the following differential equations is the L.D.E or N.L.D.E

i) $\frac{dy}{dx} = y$……………linear

ii) $(x^2 - 1)\frac{dy}{dx} = y$……linear

iii) $\frac{e^x dy}{dx} + y^2 + 4 = 0$…………..non – linear

iv) $\frac{x^2d^2y}{dx^2} = \frac{y\,dy}{dx}$ ……………non – linear

v) $\frac{x^2 dy}{dx}$ = 1 + xy………linear

vi) $(x^2 - y^2)\frac{dy}{dx}$ = 2xy……Non – linear

Formulation of a differential

Differential equation may be formed when arbitrary constant are eliminated from a given function.

Examples 1

Y = A sin x + B cos x, where A and B are two arbitrary constants

Solution

$\frac{dy}{dx}$ = A cos x – B sin x

$\frac{d^2y}{dx^2}$ = -A sin x – B cos x

This is identical to the original eqn with opposite signs

$\frac{d^2y}{dx}$ = -1(A cos x + B sin x)

$\frac{d^2y}{dx^2}$ = -y

Form a DE whose solution is the form $y = Ae^{2x} + Be^{-3X}$ Where A and B are constant.

Example 2

Form a different equation from the function $y = x + \frac{A}{x}$

Solution

$y = x + \frac{A}{x} = x + Ax^{-1}$

$\frac{dy}{dx} = 1 – Ax^{-2} = 1 - \frac{A}{x^2}$

From the given equation

$Y = x + \frac{A}{x}$

$yx = x + \frac{A}{x}$

$x(y – x) = A$

$\frac{dy}{dx} = 1- x (y – x) x^{-2}$

$= 1 - \frac{x(y-x)}{x^2}$

$\frac{x^2 dy}{dx} = x^2 – x (y – x)$

$\frac{xdy}{dx} = x – (y – x)$

$\frac{xdy}{dx} = 2x – y$

Example 3

Form the DE for $y = Ax^2 + Bx$

$Y = Ax^2 + Bx$...... (i)

$\frac{dy}{dx} = 2Ax + B$

$\frac{dy}{dx}$ - 2Ax = B……. (ii)

$\frac{d^2y}{dx^2}$ = 2A

$^1/_2 \frac{d^2y}{dx^2} = A \ldots.(iii)$

Substituting both (ii) and (iii) into (i) gives

$$Y = \left(\frac{1d^2y}{2\,dx^2}\right)x^2 + \left(\frac{dy}{dx} - 2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)x\right)x$$

$$Y = \frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} - \frac{d^2y}{dx^2}x^2$$

$$Y = \frac{x^2d^2y}{2dx^2} + \frac{xdy}{dx} - \frac{x^2d^2y}{dx^2}$$

$$Y = \frac{-x^2d^2y}{2dx^2} + \frac{xdy}{dx}$$

$\frac{x^2d^2y}{dx^2}$ – 2x $\frac{dy}{dx}$ + 2y = 0

<u>Solution to a DE</u>

This involves finding the function for which the equation is true (i.e manipulating the eqn so as to eliminate all the differential coefficients and have a relationship between x and y)

E.g. verify that the function

i) $Y = 3e^{2x}$ is a soln of DE = $\frac{dy}{dx} - 2y = 0$ for all x

ii) y (x) = sin x – cos x + 1 is a soln of eqn $\frac{d^2y}{dx} + y = 1$ for all value of x

Solution

$$\text{Let,} \quad y = 3e^{2x} \ldots\ldots (i)$$

$$\frac{dy}{dx} = 6e^{2x} \ldots\ldots (ii)$$

$$\frac{dy}{dx} = 2(3e^{2x}) \ldots\ldots iii/$$

$$\text{but} \quad y = 3e^{2x} \ldots\ldots iv/$$

Substitute equation iv into iii.

$\frac{dy}{dx} = 2y$

$\frac{dy}{dx} - 2y = 0$

<u>Direct integration</u>

Direct integration is used to solve equation which is arranged in the form $\frac{dy}{dx} = f(x)$

Examples

1. Solves $\frac{dy}{dx} = 3x^2 - 6x + 5$

Solution

$\frac{dy}{dx}$ = $3x^2 – 6x + 5$

Then $\int dy = \int (3x^2 - 6x + 5)\ dx$

= $x^3 – 3x^2 + 5x + c$

Y = $x^3 – 3x^2 + 5x + c$

2. Solve $\frac{xdy}{dx}$ = $5x^3 + 4$

Solution

Rearranging in the form $\frac{dy}{dx}$ = f (x)

$\frac{dy}{dx}$ = $5x^2 + \frac{4}{x}$

Then $\int dy = \int \left(5x^2 + \frac{4}{x}\right)$ dx

Y = $\frac{5x^3}{3}$ + 4ln x + c

Y = $\frac{5x^2}{3}$ + 4ln x + c

The solution is a called general solution since it consist a constant C (i.e. unknown constant)

3. Find the solution of the eqn $e^x \frac{dy}{dx}$ = 4 given that y = 3

When x = 0

Soln: rearrange the given equation

$= \frac{dy}{dx} = 4e^{-x} + 7$

This is called a particular soln since it contains a non – unknown variable.

First order D.E

Separating the variables

This is a method used to solve the D.E when is in the form $\frac{dy}{dx} = f(x, y)$

The variables y on the right hand-side prevents solving by direct integration

Example

1. Solve $\frac{dy}{dx} = \frac{2x}{y+1}$

Solution $\frac{dy}{dx} = \frac{2x}{y+1}$

$= (y + 1)\frac{dy}{dx} = 2x$

Integrating both sides with respect to x

$$\int (y+1)dy = \int 2xdx$$

$$\int (y+1)dy = 2\int xdx$$

$$\frac{y^2}{2} + y = x^2 + c$$

2. Solve $\frac{dy}{dx} = \frac{y^2 + xy^2}{x^2y - x^2}$

Solution

$\frac{dy}{dx} = \frac{y^2(1+x)}{x^2(y-1)}$

$\frac{(y-1)}{y^2}\frac{dy}{dx} = \frac{1+x}{x^2}$

$\int\frac{(y-1)}{y^2} dy = \int\frac{1+x}{x^2} dx$

$$\int(\frac{y}{y^2}-\frac{1}{y^2})\,dy = \int\left(\frac{1}{x^2}+\frac{x}{x^2}\right)dx$$

$$\int(\frac{1}{y}-\frac{1}{y^2})\,dy = \int\left(\frac{1}{x^2}+\frac{1}{x}\right)dx$$

$$\int\frac{1}{y}dy - \int\frac{1}{y^2}dy = \int\frac{1}{x^2}dx + \int\frac{1}{x}dx$$

$$\ln y - (-y^{-1}) = -x^{-1} + \ln x + c$$

$$\ln y + y^{-1} = -x^{-1} + \ln x + c$$

$$\ln y + y^{-1} = \frac{-1}{x} + \ln x + c$$

$$\left\{In\left(\frac{y}{x}\right) + \frac{1}{y} + \frac{1}{x} + B = 0\right\}$$

3. By separating variables solve the differential equation $(xy + x)\,dx = (x^2y^2 + y^2 + x^2 + 1)\,dy$

Solution

Given Equation;

$(xy + x)\, dx = (x^2y^2 + y^2 + x^2 + 1)\, dy$

$X (y + 1)\, dx = (y^2 (x^2 + 1) + (x^2 +1))\, dy$

$X (y + 1)\, dx = (x^2 + 1) (y^2 + 1)\, dy$

$\frac{x}{x^2+1}\, dx = \frac{y^2+1}{y+1}\, dy$

Integrating both sides

$\int \frac{x}{x^2+1}\, dx = \int \frac{y^2+1}{y+1}\, dy$

R.H.S = $\frac{y^2+1}{y+1} = y-1+\frac{2}{y+1}$

$\int \frac{x}{x^2+1}\, dx = \int \left(y - 1 + \frac{2}{y+1}\right) dy$

$\frac{1}{2} \ln (x^2 + 1) = \frac{y^2}{2} - y + 2 \ln (y + 1) + C$

First order homogenous D.E

In first order H.D.E all the terms are of the same dimension

Consider the table about dimensions

Term	X	x^2	1/x	x^n	3	y	y^n	$x^{2/}y$	x^2y
Dimension	1	2	-1	n	O	1	N	1	3

	dy/dx	dy^2/dx^2	xdy/dx	yx
	0		1	2

O shows there is no effect on the term

Which of the following equation are first order homogeneous

a) $x^2 \frac{dy}{dx} = y^2$

b) $xy \frac{dy}{dx} = x^2 + y^2$

c) $x^2 \frac{dy}{dx} = 1 + xy$

d) $(x^2 \frac{dy}{dx} = \frac{ydy}{dx}$

e) $(x^2 – y^2) \frac{dy}{dx} = 2xy$

f) $(1 + y^2) \frac{dy}{dx} = x$

Solution of 1st order Homogeneous differential equation

1st order homogeneous d.e can be written in the form $\frac{pdy}{dx} = Q$

Where both P and Q are function of and have same dimension.

Suppose p and Q hence the dimension

Then divide by x^n and use the substitution

Y = vx = v = $\frac{y}{x}$

i.e $\left[if\ y = vx \quad \frac{dy}{dx} = v + \frac{xdv}{dx}\right]$

Example 1.

Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

Solution

Since all terms are of degree 2, i.e the equation is homogeneous

Let y = vx

$\frac{dy}{dx} = v + \frac{xdv}{dx}$

And

$\frac{x^2 + y^2}{xy} = \frac{x^2 + v^2x^2}{vx^2}$

$= \frac{x^2}{x^2}\left(\frac{1+v^2}{v}\right)$

$= \frac{1+v^2}{v}$

$V + \frac{xdv}{dx} = \frac{1+v^2}{v}$

$\frac{xdv}{dx} = \frac{1+v^2}{v} - v$

$= \frac{1+v^2-v^2}{v}$

$\frac{xdv}{dx} = \frac{1}{v}$

$\frac{vdv}{dx} = \frac{1}{x}$

$\int vdv = \int \frac{1}{x} dx$

$\frac{v^2}{2}$ = lnx + c

But v = $\frac{y}{x}$

$\left(\frac{\frac{y}{x}}{2}\right)^2$ = ln x + c

$\frac{y^2}{2x^2}$ = lnx + c

$Y^{2} = 2x^2$ (lnx + c)

Example 2

Solve xy $\frac{dy}{dx} = x^2 + y^2$

Solution

All terms are of the order 2

xy $\frac{dy}{dx} = x^2 + y^2$……

Let y = vx

$\frac{dy}{dx} = v + \frac{xdv}{dx}$

From the equation…

$\frac{dy}{dx} = \frac{x^2}{xy} + \frac{y^2}{xy}$

But y = vx

$= \frac{dy}{dx} = \frac{x^2}{vx^2} + \frac{x^2 v}{vx^2}$

$= x^2 \left(\frac{1+ v^2}{x^2 v}\right)$

$= \frac{1+ v^2}{v}$

$V + \frac{xdv}{dx} = \frac{1+ v^2}{v}$

$\frac{xdv}{dx} = \frac{1+ v^2 - v}{v}$

$= \frac{1+ v^2 - v^2}{v}$

$\frac{xdv}{dx} = \frac{1}{v}$

$\int v\, dv = \int \frac{dx}{x}$

$\frac{v^2}{2} = \ln x + A$

But $v = \frac{y}{x}$

$\left(\frac{y}{x}\right)_2$ = lnx + A

$Y^2 = 2x^2$ (lnx + A)

Example 3

Solve the following

$(x^2 + xy)\frac{dy}{dx} = xy - y^2$

Solution

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2 + xy}$$

Let y = vx

$\frac{dy}{dx} = v + \frac{xdv}{dx}$

And

$\frac{xy - y^2}{x^2 + xy} = \frac{x^2v - x^2v^2}{x^2 + x^2v}$

$= \frac{v - v^2}{1+v}$

$v + \frac{xdv}{dx} = \frac{v - v^2}{1+v}$

$\frac{xdv}{dx} = \frac{v - v^2}{1+v} - v$

$= \frac{v - v^2 - v - v^2}{1+v}$

$= \frac{-2v^2}{1+v}$

$\int \frac{1+v}{-2v^2} \frac{dv}{} = \int \frac{dx}{x}$

$-\frac{1}{2}[(\int \frac{1}{v^2} dv + \int \frac{1}{v} dv)] = \int \frac{dx}{x}$

$-½ (\frac{-1}{v} + \ln v) = \ln x + c$

$$\frac{1}{2v} - \frac{1}{2}\ln v = \ln x + c$$

$$\frac{1}{v} = 2\ln x + \ln v + 2c$$

$\frac{1}{v} = \ln (x^2 V) + 2c$

But $v = \frac{y}{x}$

$\frac{x}{y} = \ln (x^2 * \frac{y}{x}) + 2c$

$\frac{x}{y} - 2c = \ln (x^2 . \frac{y}{x})$

Let $\ln A = -2c$

$\frac{x}{y} + \ln A = \ln xy$

$$\ln xy = \ln A + \frac{x}{y}$$

$Xy = Ae^{x/y}$

EXERCISE

Solve the differential equation

1. $x^2 \frac{dy}{dx}$ = y (x + y)

2. $(x^2 + y^2) \frac{dy}{dx}$ = xy

First order exact differential equation

We know that $\left[\frac{duv}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}\right]$

Now consider

$\frac{xdy}{dx} + y = e^x$

Integrating both side w.r.t.x

$= xy = e^2 + c$

$Y = \frac{1}{x}(e^x + c)$

$Y = \frac{1}{x}(e^x + C)$

2. Find the general solution of the following exact differential equation.

i) $e^x y + e^x \frac{dy}{dx} = 2$

ii) $\cos x \frac{dy}{dx} - y\sin x = x^2$

Solution

i) Given

$$e^{xy} + e^{x}\frac{dy}{dx} = 2$$

$\frac{dy}{dx}$ ($e^x y$) = 2

Integrating both sides with respect to x

$\int \frac{d}{dx}$ ($e^x y$) dx = $\int 2dx$

$e^{xy} = 2x + 4$

ii) Given

cosx $\frac{dy}{dx}$ - y sin x = x^2

$\frac{d}{dx}$ (y cos x) = x^2

Integrating both side with respect to x

= $\int \frac{d}{dx}$ ($e^x y$) dx = $\int 2dx$

Y cos x = $\frac{1x^3}{3}$ + c

3ycosx = x3 + 3C

3ycosx = x3 + 4

3. $\frac{x}{y}\frac{dy}{dx}$ + lny = x + 1

Soln

Given $\frac{x}{y}\frac{dy}{dx}$ + lny = x + 1

Integrating both sides w.r.t x

$\int \frac{d}{dx}$ (x ln y) dx = $\int (x+1)dx$

Xln y = ½ ^{x2} + x + C

2xlny + x^2 + 2x + 2c

2x lny + x^2 + x + A

Integrating factors

Consider first order differential equation of the form

$\frac{dy}{dx}$ + py =Q

Where p and Q are functions of x

Multiplying by integrating factor F both sides will make an exact equation

i.e F $\frac{dy}{dx}$ + Fpy + FQ…….i

$\frac{d\,(Fy)}{dx} = \frac{Fdy}{dx} + \frac{ydF}{dx}$

$\frac{Fdy}{dx} + \frac{ydF}{dx}$ …..ii

Comparing (i) (ii)

$\frac{Fdy}{dx}$ + Fpy = $\frac{Fdy}{dx} + \frac{ydF}{dx}$

Fpy = $\frac{ydF}{dx}$

$\frac{dF}{dx}$ = Fp

$\int \frac{1}{F}$ dF = $\int p\ dx$ by separating of variables

$$\ln F = \int Pdx$$

$$[F = e^{spdx}]$$

F = e^{spdx} is the required integrating factor

Examples

1. Solve $\frac{xdy}{dx}$ + y = x^3

Solution

$\frac{x\,dy}{dx}$ + y = x^3

$\frac{dy}{dx} + \frac{y}{x} = x^2$

Compare with $\frac{dy}{dx}$ + py = Q

P = $\frac{1}{x}$

Then $\int p\ dx = \int \frac{1}{x}$ dx

= ln x

If F = $e^{\ln x}$

F $= e^{\ln x}$

$\frac{d}{dx}$ (xy) = x^2 x

$\int \frac{d}{dx}$ (xy) = $\int x^3\, dx$

Xy = $\frac{1}{4}$ x^4 + C

2. Solve

(x + 1) $\frac{dy}{dx}$ + y + $(x + 1)^2$

Soln

(x + 1) $\frac{dy}{dx}$ + y + $(x + 1)^2$

= $\frac{dy}{dx} + \frac{y}{x+1}$ = x + 1

= $\frac{dy}{dx} + \frac{y}{(x+1)}$, y = x + 1

P = $\frac{1}{x+1}$

$\int p = \ln(x + 1)$

F = e^{sp}

= $e^{\ln (x + 1)}$

= x + 1

Y (x + 1) = $\int (x + 1)(x + 1)$

$= \int (x^2 + 2x + 1)dx$

$= \frac{1}{3}x^3 + x^2 + x + c$

$= \frac{(1+x)^3}{3} + c$

$Y = \frac{(x+1)^2}{3} + \frac{c}{3}$

3. Solve $(1 - x^2)\frac{dy}{dx} - xy = 1$

Solution $\frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$

$P = \frac{-x}{1-x^2}$

$\int pdx = \int \frac{-x}{1-x^2}dx$

$= ½ \ln (1 - x^2)$

$F = e^{\ln (1 - x^2)½}$

$= (1 - x^2)^{½}$

$Y\sqrt{1-x^2} = \int (\frac{1}{1-x^2})(\sqrt{1-x^2})dx$

$= \int \frac{1}{\sqrt{1-x^2}}dx$

$Y = \frac{\sin^{-1}x + c}{\sqrt{1-x^2}}$

4. tanx $\frac{dy}{dx}$ + y = e^x tan x

Solution

$\frac{dy}{dx} + \frac{1}{\tan x}$, y = e^x

P = $\frac{1}{\tan x}$ = cot x

$\int p dx = \int cot\ x\ dx$

= ln $\sin x$

F = $e^{\ln \sin x}$

= $\sin x$

Y $\sin x = \int e^x \sin x$ dx

Integrating by parts the R.H.S

= $\int e^x \sin x\ dx = e^x \sin x - \int e^x \cos x$ dx

= e^x sin x – (e^x cos x + $\int e^x \sin x$ dx

= $e^x \sin x - e^x \cos x - \int e^x \sin x$ dx

$2\int e^x$ sin x dx = e^x ($\sin x - \cos x$)

$\int e^x \sin x\ dx = \frac{e^x}{2}(\sin x - \cos x)$

Y sin x = $\frac{e^x}{2}(\sin x - \cos x)$ + c

Bernoulli's equation

This is a first order D.E of the form

$\frac{dy}{dx} + p(x)\,y = Q(x)\,y^n$

Where p (x) and Q (x) are functions of x or constant

Steps

$\frac{dy}{dx} + p(x)\,y = Q(x)\,y^n$……i

Divide both sides by y^n gives

$Y^{-n}\frac{dy}{dx} + p(x)\,y^{1-n} = Q(x)$…… (ii)

Let $z = y^{1-n}$

$(1-n)\,y^{-n}\frac{dy}{dx}$

Multiplying (ii) by (1 – n) both sides gives

(1 Type equation here. $- n)\,y^{-n}\frac{dy}{dx} + (1-n)\,p(x)\,y\,(1-n) = (1-n)\,Q(x)$

$\frac{dz}{dx} + (1-n)\,p(x)\,y^{(1-n)} = (1-n)\,Q(x)$

$\frac{dz}{dx} + p_1(x)\,y^{(1-n)} = Q_1(x)$

Where p_1 (x) and Q_1 (x) are functions of x or constant

But z = $y^{(1-n)}$

= $\frac{dz}{dx}$ + p_1 (x) z = Q_1 (x)……..ii

iii) is linear the use of integrating factor can be used

Example 1

Solve $\frac{dy}{dx} + \frac{1y}{x}$ = xy^2

Solution

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

Dividing both sides by y^2 gives

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Let Z = y^{-1}

$$\frac{dz}{dx} = -y^{-2}\frac{dy}{dz}$$

Multiply (ii) by -1 gives

$$-y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -x$$

$\frac{dz}{dx} - \frac{1}{x}y^{-1} = -x$

But Z = y^{-1}

$\frac{dz}{dx} - \frac{1}{x}Z = -x$

$\therefore P = -\frac{1}{x}$

$\int -\frac{1}{x}dx = -1\int \frac{1}{x}dx$

= $-lnx$

= lnx^{-1}

$\therefore From\ F = e^{\int p(x)dx}$

$\therefore F = e^{lnx}$

= x^{-1}

Then Z.F= $\int Q_1(x) f dx$

$\therefore \frac{z}{x} = \int -x\, X \frac{1}{x}dx$

= -1$\int dx$

= $-x + c$

But Z = y^{-1}

$\frac{1}{yx} = -x + c$

$\frac{1}{y} = -x^2 + cx$

$\therefore y = \left(x(c - x)\right)^{-1}$

Example 2

Solve $x^2 y - x^3 \frac{dy}{dx} = y^4 \cos x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

Solution

1st Expressing the equation in the form

$\frac{dy}{dx} + p(x)y^n$

- $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^4}{x^3} \cos x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

- $y^{-4} \frac{dy}{dx} - \frac{y^{-3}}{x} = -x^{-3} \cos x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

Let Z = y^{-3}

$$\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$-3y^{-4}\frac{dy}{dx} + \frac{3}{x}y^{-3} = +3x^{-3}\cos x$$

$$\frac{dz}{dx} + \frac{3}{x}y^{-3} = +3x^{-3}\cos x$$

But Z = y^{-3}

$$\frac{dz}{dx} + \frac{3}{x}Z = +3x^{-3}\cos x$$

$$p(x) = \frac{3}{x}$$

$$\int pxdx = \int \frac{3}{x}dx$$

$= 3lnx$

$= lnx^3$

$$\Rightarrow f = e^{lnx^3}$$

$$= x^3$$

Then

$$Z.F = \int Q(x).F dx$$

$$\Longrightarrow Z.x^3 = \int x^3. +3x^{-3}\cos x \ dx$$

$$Z.x^3 = \int 3\cos x dx$$

$$= 3\sin x + c$$

But Z = y^{-3}

$$\therefore \frac{x^3}{y^3} = 3\sin x + c$$

Example 3

Solve $y - 2x\frac{dy}{dx} = x(x+1)y^3$

Solution

Expressing the equation in the form

$$\frac{dy}{dx} + p(x)y = Q(x)y^n$$

$$\Longrightarrow \frac{dy}{dx} - \frac{1}{2x}y = -\frac{1}{2}(x+1)y^3 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.(i)$$

Then dividing by y^3

$$\Longrightarrow\ y^{-3}\frac{dy}{dx}-\frac{1}{2x}y^{-2}=\frac{-1}{2}(x+1)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

let Z = y^{-2}

$$\Longrightarrow\frac{dz}{dx}=-2y^{-3}\frac{dy}{dx}$$

Multiply (ii) by -2 gives

$$-2y^{-3}\frac{dy}{dx}+\frac{1}{x}y^{-2}=(x+1)$$

$$\Longrightarrow\frac{dz}{dx}+\frac{1}{x}y^{-2}=(x+1)$$

But Z = y^{-2}

$$\Longrightarrow\frac{dz}{dx}+\frac{1}{x}z=(x+1)$$

$$p(x)=\frac{1}{x}$$

$$\int p(x)dx=\int\frac{1}{x}dx$$

$$=lnx$$

$\Longrightarrow f = e^{lnx}$

$= x$

Then, $Z.F = \int Q(x).F dx$

$Z.x = \int (x+1).x dx$

$= \int (x^2 + x) dx$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + c$

But Z = y^{-2}

$\therefore \frac{x}{y^2} = \frac{1}{3}x^3 + \frac{1}{2}x^2 + c$

Exercise

Solve the following first order D.es

1. $2y - 3\frac{dy}{dx} = y^4 e^{3x}$

2. $x^2 \frac{dy}{dx} = 1 + xy$

Second order Differential Equations

Second order differential equation is of the form of;

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Where a, b, c are constant coefficients and $f(x)$ is a given function of x

If $f(x) = 0$, the equation is homogeneous otherwise it is a non-homogeneous

Which of the following are linear H.D.Es

1. $\frac{3dy^2}{dx} + \frac{dy}{dx} - y = 0$
2. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = \frac{e^x}{3x}$
3. $\frac{d^2y}{dx} + y = \sec x$
4. $\frac{d^2y}{dx^2} = -\frac{dy}{dx} + 6y$
5. $\frac{d^2y}{dx^2} - \frac{4dy}{dx} + 3y = 6x$

Characteristic (Auxiliary) Equation for H.DE

Consider a linear non-homogeneous 2nd order D.E

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, \qquad (a\neq 0)$$

Let y = u and y=v be two solution of the equation

Where u and v a functions of x

$$a\frac{d^2u}{dx}+b\frac{du}{dx}+cu=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$$

$$a\frac{d^2v}{dx^{\ 2}}+b\frac{dv}{dx}+cv=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$$

Adding (i) & (ii) gives

$$a\left(\frac{d^2u}{dx^2}+\frac{d^2v}{dx^2}\right)+b\left(\frac{du}{dx}+\frac{dv}{dx}\right)+c(u+v)=0\ldots\ldots\ldots(iii)$$

Then,

$\frac{d^2}{dx^{\ 2}}(u+v)=\frac{d^2u}{dx^2}+\frac{d^2v}{dx^{\ 2}}$ and

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

Becomes

$$a\frac{d^2}{dx^2}(u+v) + b\frac{d}{dx}(u+v) + c(u+v) = 0$$

If y = u and y = v are the solutions of the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

Also, $\boldsymbol{y = u + v}$ is a solution

Suppose a = 0

$$\Rightarrow b\frac{dy}{dx} + cy = 0$$

i.e. $\frac{dy}{dx} + \frac{c}{b}y = 0$

$$\Rightarrow \frac{dy}{dx} + ky = 0 \left(where\ k = {}^{c}/_{b}\right)$$

$\frac{dy}{dx} = -ky$ (separable)

$\int \frac{dy}{y} = -k \int dx$

$lny = -kx + c$

$\therefore y = e^{-kx}.e^{c}$

$y = e^{-kx}$ (where e^{c} is constant)

Take m for –k

- $y = e^{mx}$ is the solution of dy +ky

Also will be the solution of the equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$ if it satisfies the equation

If $y = Ae^{mx}$

$\frac{dy}{dx} = Ame^{mx}$

$\frac{d^2y}{dx^2} = Am^2e^{mx}$

$\Longrightarrow aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0$

$\Longrightarrow (am^2 + bm + c)Ae^{mx} = 0$

$[am^2 + bm + c = 0]$ *called auxiliary equation*

The values will be $m = m_1$ and $m = m_2$

If $y = Ae^{m_1x}$ *and* $y = Be^{m_2x}$ are two solution

$\Longrightarrow y = Ae^{m_1x} + Be^{m_2x}$

Note that:

If the Auxiliary equation has

(i) Two real roots m_1 *and* m_2 $(b^2 - 4ac > 0)$

The solution is $y = Ae^{m_1x} + Be^{m_2x}$

(ii) Equal roots $(i.e.\ m_1 = m_2)$

$\Longrightarrow (b^2 - 4ac = 0)$

The solution is $y = e^{m_1x}(A + Bx)$

(iii) 2 Complex roots to the auxiliary equation

$$p \pm qi \; (b^2 - 4ac = 0)$$

The solution is $y = e^{px}(C_1 \cos qx + C_2 \sin qx)$

Examples

Solve the following 2[nd] order Des

1. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 9y = 0$

Solution

Auxiliary equation $m^2 + 5m + 6 = 0$

$$\Rightarrow m^2 + 5m + 6 = 0$$

$$(m + 2)(m + 3) = 0$$

$$\therefore m = -2 \; or \; m = \; -3$$

$$\therefore Solution\, is\; y = Ae^{-3x} + Be^{-2x}$$

2. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Solution

The auxiliary equation is

$m^2 + 6m + 9 = 0$

$(m + 3)(m + 3) = 0$

$\therefore m = -3 \; or \; m = \; -3$

$\therefore solution \; is \; y = \; e^{-3x}(A + Bx)$

3. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 9y$

<u>Solution</u>

Auxiliary equation is

$m^2 + 6m + 9 = 0$

$\therefore m = \frac{-4 \pm \sqrt{16-36}}{2}$

$\frac{-4 \pm \sqrt{-20}}{20}$

$m = 2 \pm \sqrt[i]{5}$

In this case P=-2 and q = $\sqrt{5}$

$$\therefore y = e^{-2x}\left(C_1 \cos\sqrt{5x} + C_2 \sin\sqrt{5x}\right)$$

Exercise

Solve the following

1. $\frac{d^2y}{dx} - 12\frac{dy}{dx} + 36y = 0$

2. $\frac{2d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$

3. $\frac{d^2y}{dx} + 7 = 0$

<u>Non homogeneous 2nd order D.E</u>

Consider the equation

$$a\frac{d^2y}{dx} + b\frac{dy}{dx} + cy = f(x) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(i)$$

If $f(x) = 0, the\ auxiliary\ equation\ is\ am^2 + bm + c = 0$

$m = m_1\ and\ m = m_a\ and\ the\ solution\ is$

$y = Ae^{m_1x} + Ae^{m_2x}$

General solution = complementary function +particular integrate

The general solution of (i) is given by

Note that

$y = Ae^{m_1 x} + Be^{m_2 x}$ is called complementary function matas R.H.S Zero

Y = f (x) is called particular intergral makes R.H.S $\neq 0$

Consider the R.H.S function

i.e if $f(x) = K$ assume $y = C_1$

$f(x) = Kx$ $y = C_1 x + C_2$

$f(x) = kx^2$ $y = c_1 x^2 + c_2 x + c_3$

$f(x) = k \sin x \, or \, k \cos x$ $y = c_1 \cos x + c_2 \sin x$

$f(x) = k \sinh x \;\, or \, k \cos hx$ $y = c_1 \cosh x + c_2 \sinh x$

$f(x) = e^{kx}$ $y = ce^{kx}$

Examples

1. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 24$

Solution

C.F solve L.H.S = 0

$$m^2 + 5m + 6 = 0$$

$$m^2 - 3m - 2m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$\therefore \ m_1 = 2 \ and \ m_2 = 3$$

$$y = Ae^{2x} + Be^{3x}$$

$$P.I \ f(x) = 24$$

Assume $y = c$

$$\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = 0$$

Substituting in the given equation gives

$0 - 5(0) + 6c = 24$

$\therefore P.I = 4$

$\therefore General\ solution\ is\ y = C.F + P.I$

$= Ae^{2x} + Be^{3x} + 4$

2. Solve $\frac{d^2y}{dx} - 3\frac{dy}{dx} + 2y = x^2$

Solution

(i) C.F: $m^2 + 3m + 2 = 0$

$(m-1)(m-2) = 0$

$\therefore\ m_1 = 1\ and\ m_2 = 2$

$y = Ae^x + Be^{2x}$

(ii) $P.I: Assume\ y = c_1x^2 + c_2x + c_3$

$\frac{dy}{dx} = 2c_1x + c_2$

$$\frac{d^2y}{dx^2} = 2c_1$$

Substituting into the given equation

$$2C_1 - 3(2C_1x + C_2) + 2(C_1x^2 + C_2x + C_3 = X^2)$$

$$2C_1 - 6C_1x - 3C_2 + 2C_1x^2 + 2C_2x + 2C_3 = x^2$$

Collecting like terms

$$\Longrightarrow 2C_1x^2 + (2C_2 - 6C_1)x + (2C_1 - 3C_2 + 2C_3) = x^2$$

Comparing L.H.S to R.H.s

$$C_1 = \frac{1}{2}$$

$$2C_2 - 6C_1 = 0$$

$$2C_2 = 6C_1 \qquad \Longrightarrow \quad C_2 = 3C_1 \quad \Longrightarrow C_2 = \frac{3}{2}$$

$$2C_1 - 3C_2 + 2C_3 = 2\ x\ \frac{1}{2} - 3\ x\ \frac{3}{2} + 2\ x\ C_3 = 0$$

$$= 1 - \frac{9}{2} + 2C_3 = 0$$

$$= -\frac{7}{9} + 2C_3 = 0$$

$2C_3 = \frac{7}{2}$

$C_3 = \frac{7}{4}$

$P.I \implies y = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}$

$\therefore$ General solution

$y = Ae^x + Be^{2x} + \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}$

3. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 10\cos x$

Given that $y = 1, \frac{dy}{dx} = 0 \; when \; x = 0$

Solution

C.F: $m^2 + 3m + 2 = 0$

$m^2 + m + 2m + 2 = 0$

$m_1 = -1 \; and \; m_2 = -2$

$\therefore y = Ae^{-x} + Be^{-2x}$

P.I: Assume $y = C_1 \cos x + C_2 \sin x$

$\frac{dy}{dx} = -C_1 \sin x + C_2 \cos x$

$\frac{d^2y}{dx^2} = -C_1 \cos x - C_2 \sin x$

$\Longrightarrow -C_1 \cos x - C_2 \sin x + 3(-C_1 \sin x + C_2 \cos x\,)_{+2}(C_1 \cos x + C_2 \sin x\,)$

$3C_2 \cos x + C_1 \cos x + C_2 \sin x - 3C_1 \sin x = 10 \cos x$

$(3C_2 + C_1) \cos x + (C_2 - 3C_1) \sin x = 10 \cos x$

$\Longrightarrow 3C_2 + C_1 = 10 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

$C_1 = 10 - 3C_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

$C_2 - 3C_1 = 0 \Longrightarrow\ C_2 = 3C_1\ i.e.\ C_1 = \frac{C_2}{3}$

Substitute into (i)

$\frac{C_2}{3} = 10 - 3C_2$

$C_2 = 30 - 9C_2$

$10C_2 = 30$

$C_2 = 3$

$C_1 = \frac{3}{3} = 1.$

$y = Ae^{-x} + Be^{-2x} + \cos x + 3\sin x$

But y = 1, $\frac{dy}{dx} = 0 \; when \; x = 0$

1= A + B + 1

A + B = 0

$\frac{dy}{dx} = -Ae^{-x} - 2Be^{-2x} + 3\cos x - C_1 \sin x$

$0 = -$A-2B+3

$A + 2B = 3 \ldots$..............................(i)

$(ii) - (i) \; gives$

B = 3

Substituting into (ii) gives

A + 3 = 0

A = -3

$$\therefore y = -3e^{-x} + 3e^{-2x} + 3\sin x - 3\cos x$$

4. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x$.

Solution

C.F: $m^2 + 5m + 6 = 0$

$(m-2)(m-3) = 0$

$\therefore m_1 = 2 \; and \; m_2 = 3$

$y = Ae^{2x} + Be^{3x}$

$P.I$

Let y = Ce^x

$\frac{dy}{dx} = Ce^x$

$\frac{d^2y}{dx^2} = Ce^x$

$Ce^x - {}_5Ce^x + 6Ce^x = e^x$

${}_2Ce^{2x} = e^x$

2C=1

$\therefore C = 1/2$

$\therefore General\,solution\quad y = Ae^{2x} + Be^{3x} + \frac{1}{2}e^{x}$

Note that: if P.I is contained in the C.F multiply the assumed P.I by x and go on

Example

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 3e^{-2x.}$

Solution

C.F: $m^2 - 2m - 8 = 0$

$m^2 + 2m - 4m - 8 = 0$

$(m+2)(m-4) = 0$

$\therefore\ m_1 = -2\ and\, m_2 = 4$

$y = Ae^{-2x} + Be^{4x}$

P.I: Assume y =Ce^{-2x}

Since e^{-2x} is already contained cxe^{-2x}

$\frac{dy}{dx} = ce^{-2x} + (cx)x - 2e^{-2x}$

$= Ce^{-2x} - 2cxe^{-2x} = Ce^{-2x}(1-2x)$

$\frac{d^2y}{dx^2} = -2Ce^{-2x} - 2Ce^{-2x}(1-2x)$

$= Ce^{-2x}(-2-2+4x)$

$= Ce^{-2x}(4x-4)$

$Ce^{-2x}(4x-)4) - 2\left(Ce^{-2x}(1-2x)\right) - 8xe^{-2x} = 3e^{-2x}$

$Ce^{-2x}(4x-4-2+4x-8x) = 3e^{-2x}$

$-6C = 3$

$c = -\frac{1}{2}$

$\therefore P.I\ is\ y = -\frac{1}{2}xe^{-2x}$

$general\ solution\ y = Ae^{-2x}$ +B $e^{4x} - \frac{1}{2}xe^{-2x}$

Exercise

Solve the following

(i) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=e^x$

2nd order equations which are reducible to 1st order

Consider 2nd order equation which can not be written in the form

$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

i.e $\frac{x^2d^2y}{dx^2}-2x\frac{dy}{dx}=0$

Such equation will be solved by the substitution of: -

$$\left[\frac{dy}{dx}=P\right]$$

$$\Longrightarrow \frac{dp}{dx}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{dp}{dx}=\frac{d^2y}{dx^2}$$

$$\frac{x^2dp}{dx^2}-2p=0$$

$$\int\frac{dp}{p}=\int\frac{2dp}{x}$$

$$lnp=2lnx+c$$

$$lnp=2lnx^2+c$$

$$lnp=lnAx^2$$

$$p = Ax^2$$

$$\therefore \frac{dy}{dx} = Ax^2$$

$$\int dy = \int Ax^2 dx$$

$$y = \frac{Ax^3}{3} + B$$

Where A and B are constants

Example

Solve $(1 + x^2)\frac{d^2y}{dx^2} = 2x\frac{dy}{dx}$

Solution

Let $P = \frac{dy}{dx}$

$$(1 + x^2)\frac{dp}{dx} = 2xp$$

$$\frac{dp}{p} = \frac{2x}{1+x^2}\text{dx}$$

$$\int \frac{1}{p} dp = \int \frac{2x}{1+x^2}\text{ dx}$$

$$lnp = ln(x^2 + 1) + c$$

$$lnp = lnA(x^2 + 1)\ (c = lnA)$$

$$p = A(x^2 + 1)$$

$$\frac{dy}{dx} = A(x^2 + 1)$$

$$\int dy = \int A(x^2 + 1)dx$$

$$y = A\left(\frac{x^3}{3} + x\right) + B$$

$$\therefore y = A\left(\frac{x^3}{3} + x\right) + B$$

Example

Solve $\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2$

Solution

Let p = $\frac{dy}{dx}$

$$\frac{dp}{dx} = 2(p)^2$$

$$\frac{dp}{dx} = 2p^2$$

$$\int \frac{dp}{p^2} = \int 2dx$$

$$\frac{-1}{p} = 2x + c$$

$$p = \frac{-1}{2x+c}$$

$$\frac{dy}{dx} = \frac{-1}{2x+c}$$

$\int dy = -1 \int \frac{-1}{2x+c}\, \mathrm{dx}$

$y = -\frac{1}{2} ln(2x + c) + A$

$\therefore y = -\frac{1}{2} ln(2x + c) + A$

Example

$\frac{y\, d^2 y}{dx} = \left(\frac{dy}{dx}\right)^2$

Solution

Let $p = \frac{dy}{dx}$

$y\left(\frac{dp}{dx}\right) = p^2$

But, $\frac{dp}{dx} = \frac{dy}{dx} \cdot \frac{dp}{dy}$

$\frac{dp}{dx} = p.\frac{dp}{dy}$

$y\left(p.\frac{dp}{dy}\right) = p^2$

$\frac{1}{p} dp = \frac{1}{y} dy$

$$lnp = lny + c$$

In p= ln Ay where c= ln A

P = Ay

i.e. $\frac{dy}{dx} = Ay$

$$\int \frac{1}{y} dy = \int Adx$$

$$lny = Ax + B$$

$$\therefore y = e^{Ax+B}$$

Note that:

- If P.I is contained in C.F, and C.F is of real roots Assume $y = cx^2e^{kx}$ if $f(x) = e^{kx}$

- If C.F is a distinct root, but one root is the same as that on P.I, assume $y = cxe^{kx}$ if $f(x) = e^{kx}$

- If $f(x) = e^{k_1x} + e^{k_2x}$, assume $y = e^{k_1x}$ independently followed by $y = e^{k_2x}$ then add to obtained a P.I

Applications of differential Equations

1. i. Population growth

The rate of population growth id differently proportional to the number of inhabitants present at specific time.

If N is the number of inhabitants at specific time.

$$\frac{dN}{dt} \propto N$$

$$\left[\frac{dN}{dt} = KN\right]$$

$$\int \frac{dN}{N} = \int kdt$$

$$lnN = kt + c$$

When $t = 0 \Longrightarrow N = N_o$

i.e. $lnN_o = k(o) + c$

$$lnN_o = c$$

$$\therefore c = lnN_o$$

$$\Longrightarrow = kt + c$$

$$lnN = kt + lnN_o$$

$$lnN - lnN_o = kt$$

$$ln\left(\frac{N}{N_o}\right) = kt$$

$$\Rightarrow \left[\frac{N}{N_o} = e^{kt}\right]$$ growth equation

$$also\ [N = N_o e^{kt}]$$

$example$

If the population of a certain place doubles in 50 years. After how many years will the population treble and the assumption that the rate of increase is proportional to the number of inhabitant in the place.

Solution

Let N be the number of inhabitants at time t.

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = KN$$

$$\int \frac{dN}{N} = \int kdt$$

$$lnN = kt + c$$

When $t = 0 \Rightarrow N = N_o$

$lnN_o = k(o) + c$

$lnN_o = c$

$\therefore c = lnN_o$

$\Longrightarrow = kt + c$

$lnN = kt + lnN_o$ ………………………..(i)

When t = 50 years, N=2N_o

$In\ (2No) = k(50) + \ In\ No$

$ln(2N_o) - lnN_o = k(50)$

$ln\frac{2N_o}{N_o} = 50k$

$ln(2) = 50k$

K = $\frac{1}{50}ln2$

Substituting into (i)

ln N = kt +ln N_o

$$lnN = \left(\frac{1}{50}ln2\right)t + lnN_o \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$

$$when\, t = t,\ N = 3N_o$$

$$substituting\ into\ (ii) gives$$

$$ln(3N_o) = \left(\frac{1}{50}ln2\right)t + lnN_o$$

$$ln(3N_o) - ln(N_o) = \left(\frac{1}{50}ln2\right)t$$

$$ln\left(\frac{3N_o}{N_o}\right) = \left(\frac{1}{50}ln2\right)t$$

$$ln3 = \left(\frac{1}{50}ln2\right)t$$

$$t = \frac{50ln3}{ln2}$$

$$t = 79.2$$

$$\therefore t \approx 79\, years$$

Example

The population of Kenya is known to increase to a rate proportional to the number of people living there at any given time. In 1986 the population was 1.1 times that of 1984 and 1987 the population was 18,000,000. What was the population in 1984?

Solution

Let N be the number of people at time, t

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = KN$$

$$\int \frac{dN}{N} = \int k dt$$

$$lnN = kt + c$$

NUMERICAL METHOD

Introduction

Numerical methods can be used to find roots of a function

→We find roots of a function by;

1. Direct method

2. Iterative method

ERROR

An error can be defined as the *deviation from accuracy* or correctness

Error $\Delta_x = x - x_0$

Where;

X = is the exact value of a number

X_0 = is an approximate value of a number

Example

If X_0 = 3.14 and x= π then

$\Delta_x = x - x_0$

$= \pi - 3.14$

=0.001592654

TYPES OF ERROR

A) Systematic error

This is a predictable error or constant caused by imperfect calibration of measurements instruments or something is wrong from the measuring instrument

B) Random error

Unpredictable error caused either by weather or anything else.

Sources of errors

1. Experimentation error/modeling error

2. Truncation error/terminating error

E.g. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

You can see that the series is terminated at power of 3

3. Rounding off numbers

4. Mistakes and blunder

ABSOLUTE AND RELATIVE ERROR

Absolute error

Is the difference between the measured value of a quantity X_0 and its value

Absolute error $= \quad /X - X_0/=/\Delta X/$

Relative error

$is\ the\ ratio\ between\ error\ |x - x_0|\ to\ the\ exact\ value\ of\ x$

i.e. Relative error = $\frac{X-X_O}{X} = \frac{\Delta X}{X}$

A relative error gives an indication of how good measurement is relative to the size measurement is relative to the size of the thing that measured.

Example

$If\ x_0 = 3.14\ and\ x = \pi\ find$

i) Absolute error

ii) Relative error

iii)Percentage error

Solution

i. Absolute error = $/X - X_0/$

$\pi = 3.14$

$= X - X_0$

âˆ†$_x$ = 0.001592654

ii. Relative error $= \frac{/\Delta x/}{x}$

$= \frac{0.001592654}{\pi}$

= 0.000050696(9dp)

iii. Percentage error $= \frac{/\Delta x/}{x} \times 100\%$

= 0.000050696 x 100%

= $5.0696 \times 10^{-2}\%$

<u>Roots by iterative methods</u>

Iterative method is used to find a root of function by approximations repeatedly.

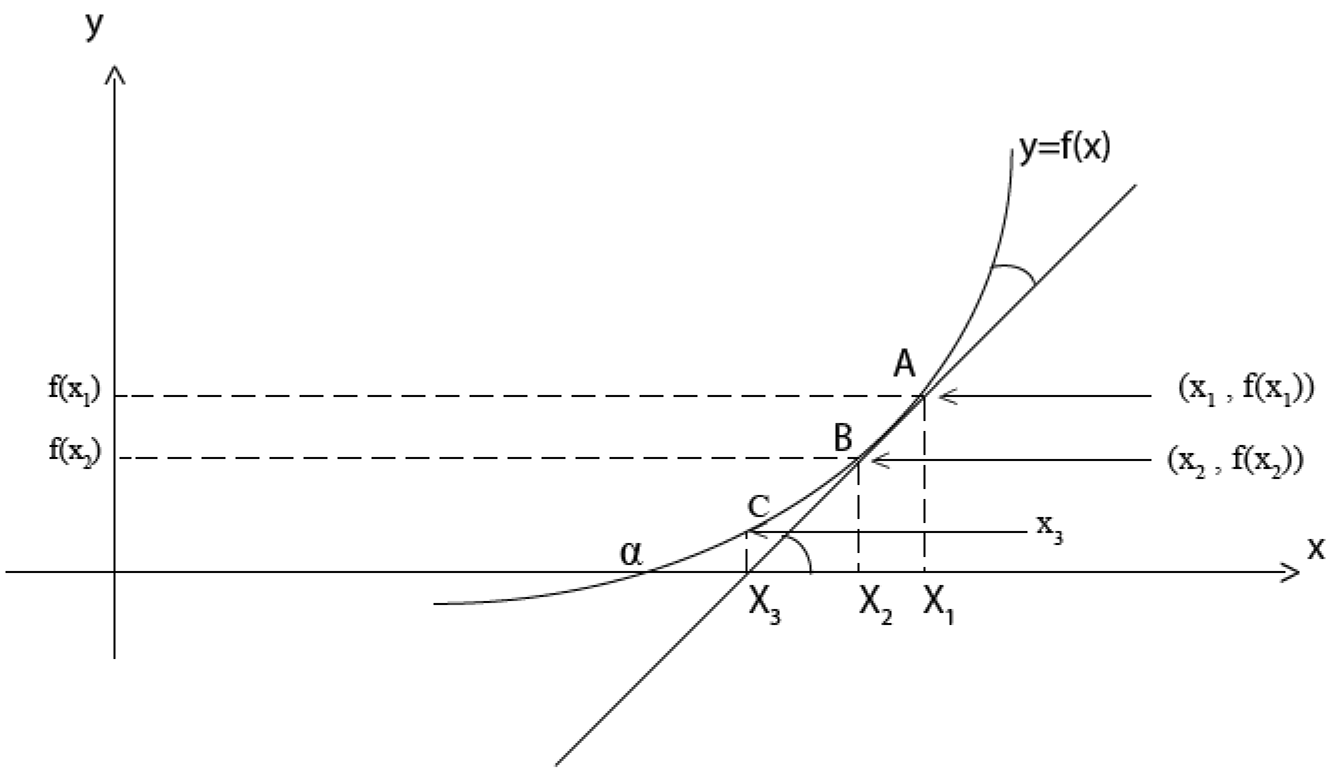
If $f(x_1), f(x_2) < 0$ the root lies between X_1 and X_2

Newton's Raphson formula

The formula is based on the tangent lines drawn to the curves through x-axis

Consider the graph below

Suppose f(x) = 0 has a root $\propto$ that x is an approximation for $\propto$



Choosing a point which is very close to ï„ƒ let be X_1

$X_1 \rightarrow \propto$

$\rightarrow f(x_1) \approx f(\propto) = 0$

A line AB is drawn tangent to the curve at a point A where $A(X_1, f(X_1))$

X_2 is the point which is very near to $\propto$ *compared to* X_1

$$Slope\overline{BC} = f^1(x_2) = \frac{f(x_2 - 0)}{x_2 - x_3}$$

This slope is equal to the tangent of the curve at $X=X_1$

i.e. $f'(x) = \frac{f(x_1)-f(x_2)}{X_1-X_2}$

But At B $f(X_2) = 0$

$$f'(x) = \frac{f(x_1) - 0}{X_1 - X_2}$$

$$X_1 - X_2 = \frac{f(x_1)}{f'(x)}$$

$$X_2 = X_1 - \frac{f(x_1)}{f'(x)}$$

X_3 will be the best approximation

$$slope\,\overline{BC} = f'(x_2) = \frac{f(x_2 - 0)}{x_2 - x_3}$$

$$x_2 - x_3 = \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In general N-R formula can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example

Show that the equation $X^3 - X^2 + 10x - 2 = 0$ has a root between x = 0 and x = 1 and find the approximation for this root by carrying out 3 iterations

Solution

$$x_1 = 0 \rightarrow f(0) = -2$$

$$x_2 = 1 \rightarrow f(1) = 8$$

$f(x_1)f(x_2) < 0 \rightarrow root(0,1)$

$choose\ any\ value\ in\ (0,1) for\ our\ first\ approximation$

$ie\ let\ x = 0.5$

$given\ f(x) = x^3 - x^2 + 10x - 2$

$then,$

$f'(x) = 3x^2 - 2x + 10$

$f(x_1) = 2.875\ , f'(x_1) = 9.75$

$$\rightarrow x_2 = 0.5 - \frac{2.875}{9.75}$$

$= 0.2051 (to\ 4\ decimal\ places) \rightarrow first\ iteration$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.2051 - \frac{0.0178358}{9.6738988}$$

$= 0.2033\ (to\ 4dp)(second\ iteration)$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 0.2033 - \frac{0.0000789}{9.7174048}$$

$x_4 = 0.2033 (to\ 4dp)$

Application of N-R Formula

1. Find approximation for roots of numbers suppose we want to approximate $\sqrt{N}$

$let\ x = \sqrt{N}$

$x^2 = N$

$x^2 - N = 0$

$let\ f(x) = x^2 - N$

$\rightarrow f'(x) = 2x$

$given\ approximation\ x_n$
$\rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{(x_n{}^2 - N)}{2x_n}$$

$$= \frac{2x_n{}^2 - x_n{}^2 + N}{2x_n}$$

$$= \frac{x_n{}^2 + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Example

B y using 2 iteration only and starting with an initial value 2, find the square root of 5 correct to four decimal place

Solution

Let x=$\sqrt{5}$

X^2=5

$X^2 - 5 = 0$

Let $f(x) = x^2 - 5$ by N.R formula

$f'(x) = 2x$

Given $X_1 = 2$

Then

First iteration

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= x_1 - \frac{(x_1{}^2 - 5)}{2x_1}$$

$$= \frac{2x_1{}^2 - x_1{}^2 + 5}{2x_1}$$

$$\frac{x_1{}^2 + 5}{2x_1}$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right)$$

$$x_2 = \frac{1}{2}\left(2 + \frac{5}{2}\right)$$

$$= \frac{1}{2} \times \frac{9}{2}$$

$$= \frac{9}{4}$$

$second\ iteration$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{5}{x_2}\right)$$

$$=\frac{1}{2}\left(\frac{9}{4}+5\times\frac{9}{4}\right)$$

$$=\frac{1}{2}\left(\frac{81+80}{36}\right)$$

$x_3 = 2.25$

Example

Apply the N.R formula to establish the root of a number A

Solution

$let\ x = \sqrt[r]{A}$

$x^r = A$

$x^r - A = 0$

$let\ f(x) = x^r - A$

$f'(x) = rx^{r-1}$

$from\ N.R\ fomula$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$for\ given\ x_n$

$$x_{n+1} = x_n - \frac{(x_n{}^r - A)}{rx_n{}^{r-1}}$$

$$=\frac{x_n\left(rx_n{}^{r-1}\right) - x_n{}^r + A}{rx_n{}^{r-1}}$$

$$=\frac{rx_n{}^{r+1} - x_n{}^{r+1} + Ax_n}{x_n{}^r}$$

$$x_{n+1} = (r-1)x_n + \frac{Ax_n}{x_n{}^r}$$

Finding approximations for reciprocals of numbers

Suppose we want to approximate $\frac{1}{N}$

$$let\ x = \frac{1}{N}$$

$$\rightarrow N = \frac{1}{x}$$

$$N - \frac{1}{x} = 0$$

$$let\ f(x) = N - \frac{1}{x}\ and\ using\ N.R\ method$$

$$f'(x) = \frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{\left(N - \frac{1}{x}\right)}{\frac{1}{{x_n}^2}}$$

$$= x_n - \left(\frac{x_nN - 1}{x_n}\right){x_n}^2$$

$$= x_n - ({x_n}^2N - x_n)$$

$$= x_n - {x_n}^2N + x_n$$

$$= 2x_n - {x_n}^2N$$

$$x_{n+1} = 2x_n - {x_n}^2N$$

Example

Use N.R formula to find the inverse of 7 to 4, and perform 3 iteration only starting with X_o=0.1

Solution

$$let\ x = \frac{1}{7}$$

$$7 = \frac{1}{x} \rightarrow 7 - \frac{1}{x}$$

$let\ f(x) = 7 - \frac{1}{x}$

$f'(x) = \frac{1}{x^2}$

$by\ N.R\ fomula$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\ \ given\ x_o = 0.1$$

$$x_n = x_o - \frac{f(x_o)}{f'(x_o)}$$

$$= x_o - \frac{\left(7 - \frac{1}{x_o}\right)}{\frac{1}{{x_o}^2}}$$

$$= x_o - \left(7 - \frac{1}{x_o}\right).{x_o}^2$$

$$= x_o - \left(\frac{7x_o - 1}{x_o}\right) \times {x_o}^2$$

$$= x_o - (7x_o - 1)x_o$$

$$= x_o - 7{x_o}^2 + x_o$$

$$= 2x_o - 7{x_o}^2$$

$$x_1 = 2(0.1) - 7(0.1)^2$$

$$= 0.13$$

$$x_2 = 2(0.13) - 7(0.13)^2$$

$= 0.1417 \ (to\ 4\ dp)$

$x_3 = 2x_2 - 7(x_2)^2$

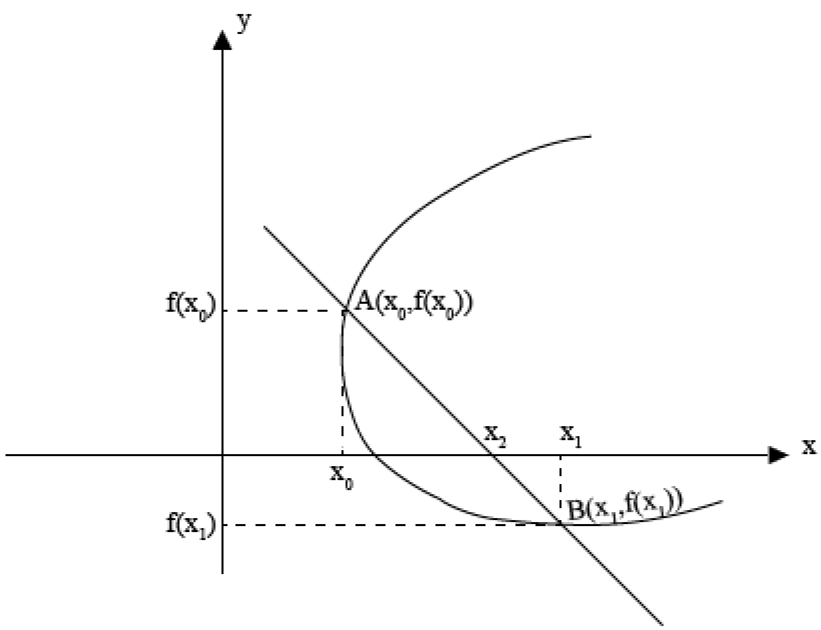
$= 2 \times 0.1417 - 7(0.1417)^2$

$= 0.142847777$

$\rightarrow x_3 = 0.1428$

SECANT METHOD

The secant method requires two initials values X_0 and X_1



Line AB is a secant line on the curve f(x)

$Considers\ points\ A(x_0, f(x_0))and\ B(x_1, f(x_1))equation\ of\ line\ AB$

$$y = \frac{f(x_1) - f(x_o)}{x_1 - x_o}(x - x_1) + f(x_1)$$

We find the roots of this, the value of x such as that y=0

$$\rightarrow 0 = \frac{f(x_1) - f(x_o)}{x_1 - x_o}(x - x_1) + f(x_1)$$

$$x - x_1 = \frac{-(x_1 - x_o)f(x_1)}{f(x_1) - f(x_o)}$$

$$x = x_1 - \frac{x_1 - x_o}{f(x_1) - f(x_o)}.f(x_1)$$

$taking\ x = x_2$

$$x_2 = x_1 - \left(\frac{x_1 - x_o}{f(x_1) - f(x_o)}\right) \times f(x_1)$$

$x = x_3$

$$x_3 = x_2 - \left(\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right) \times f(x_2)$$

In general secant formula is given

$$x_n = x_{n+1} - \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.f(x_{n-1})$$

Comparison with Newton's method

-Newton's converges faster (order 2 against ≈1.6)

- Newton's requires the evaluation of f and f_1 at every step

-Secant method only requires the evaluation of f

Example

Calculate in 3 iteration the root of the function f(x)= x^2-4x+2 which his between X_0=0 and X_1=1

Solution

$$x_o = 0 \rightarrow f(x_o) = 2$$

$$x_1 = 1 \rightarrow f(x_1) = -1$$

$from\ secant\ method$

$$x_n = x_{n+1} - \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.f(x_{n-1})$$

$$x_2 = x_1 - \frac{x_1 - x_o}{f(x_1) - f(x_o)}.f(x_1)$$

$$= 1 - \frac{1 - 0}{-1 - 2} \times (2) = \frac{2}{3}$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)}.f(x_2)$$

$f(x_1) = -1$

$f(x_2) = \left(\frac{2}{3}\right)^2 - 4 \times \frac{2}{3} + 2 = -\frac{9}{2}$

$\rightarrow x_3 = \frac{2}{3} - \left(\frac{\frac{2}{3} - 1}{-\frac{9}{2} + 1}\right) \times -\frac{9}{2} = \frac{4}{7}$

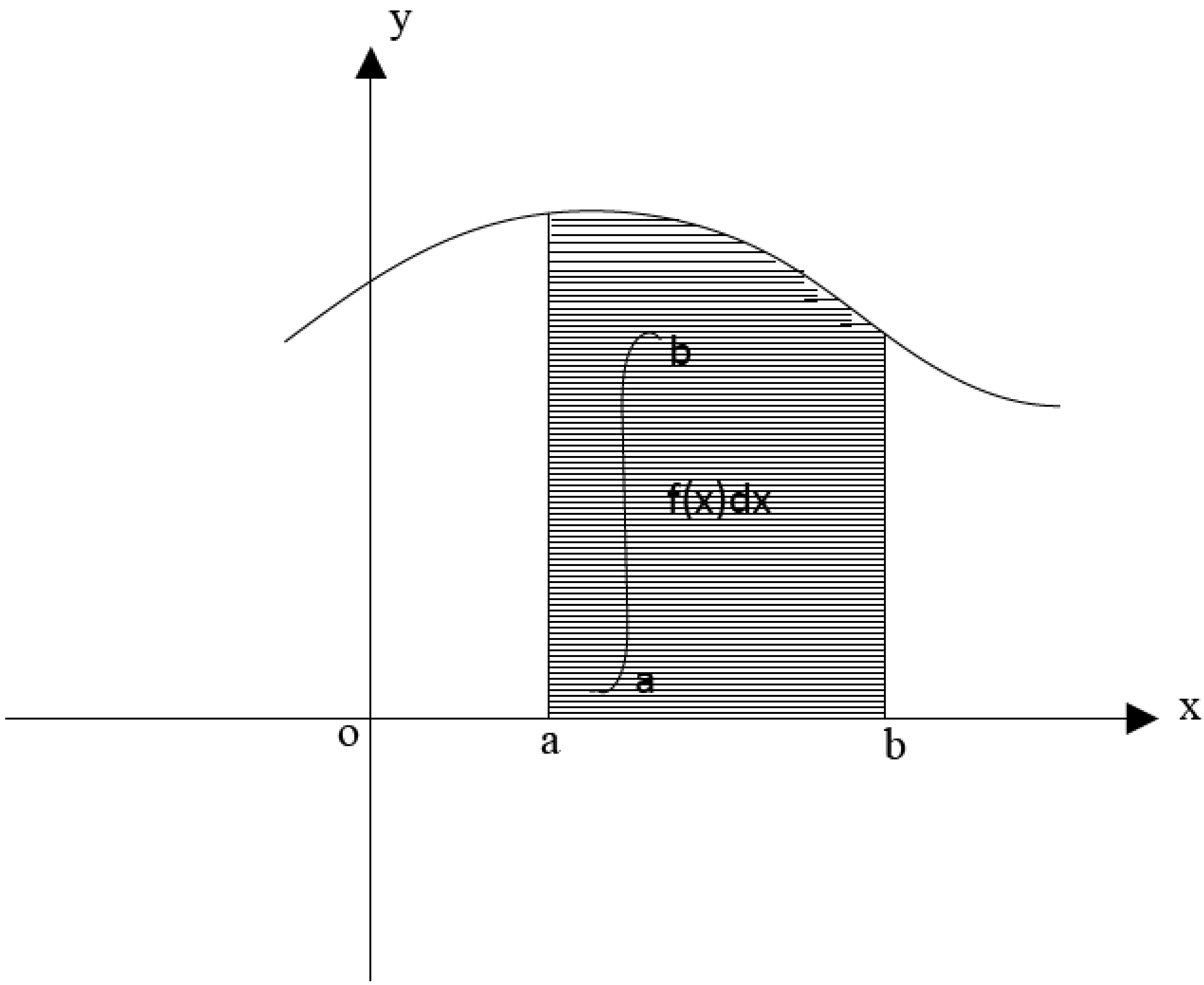
$f(x_3) = \left(\frac{4}{7}\right)^2 - 4\left(\frac{4}{7}\right) + 2 = \frac{2}{49}$

$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} . f(x_3)$

$= \frac{4}{7} - \frac{\frac{4}{7} - \frac{3}{2}}{\frac{2}{49} + \frac{9}{2}} \times \frac{2}{49}$

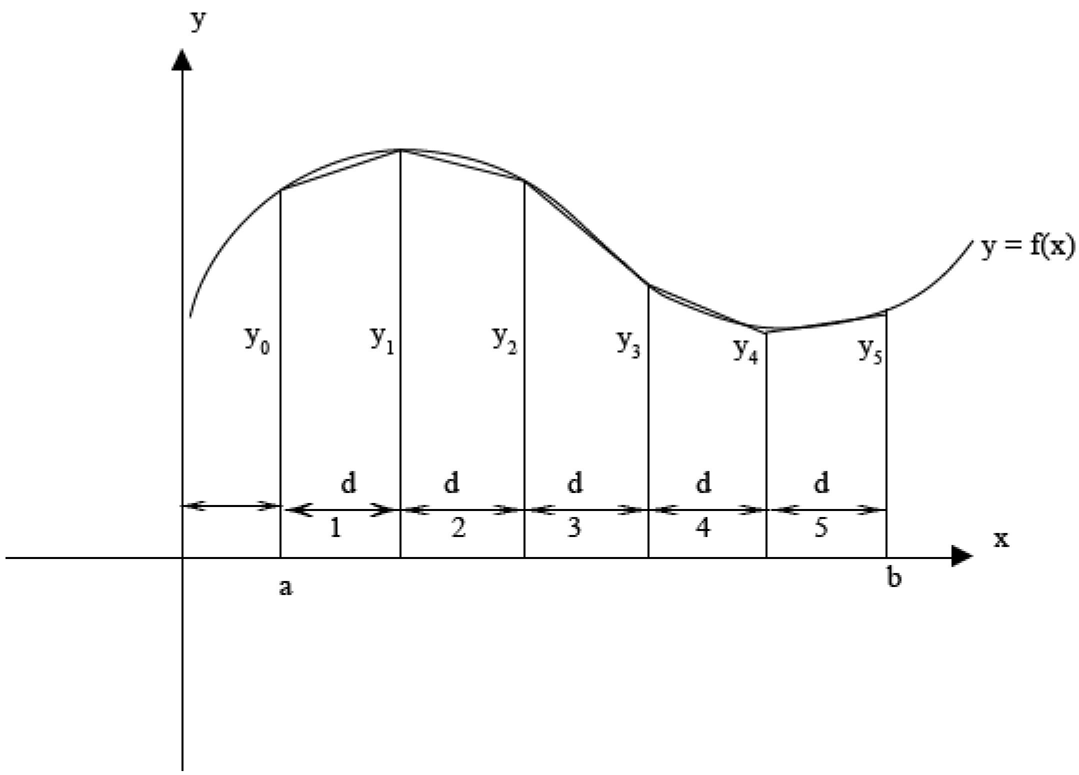
NUMERICAL INTEGRATION

Definite integral $\int_a^b f(x)dx$ is used to determine the area between y= f(x), the x -axis and the ordinates x = a and x = b



An approximate value for the integral can be found by estimating this area by another two methods

A. <u>Trapezium rule</u>



$$area\ of\ trapezium = \frac{1}{2}d(y_o + y_1)$$

adding the area of these steps

$$\int_a^b f(x)dx \approx \frac{1}{2}d(y_o + y_1) + \frac{1}{2}d(y_1 + y_2) + \frac{1}{2}d(y_2 + y_3) + \frac{1}{2}d(y_3 + y_4)$$

$$\approx \frac{1}{2}d(y_o + 2y_1 + 2y_2 + 2y_3 + y_4)$$

$$\frac{1}{2}d(y_o + y_4 + 2(y_1 + y_2 + y_3))$$

note thet the width of all strips must be the same

in general

$$\int_a^b f(x)dx \approx \frac{1}{2}d(y_o + y_4 + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})$$

$$d = \frac{b-a}{n-1}\ if\ n\ is\ the\ number\ of\ ordinates$$

$, where\ n-1\ is\ the\ number\ of\ strips,$

$n\ is\ the\ number\ of\ ordinates,$

$$d=\frac{b-a}{n}\ if\ n\ is\ the\ number\ of\ strips$$

Example

Estimate to 4 decimal places $\int_0^1 \frac{1}{1+x^2}dx$

Using five ordinates by the trapezium rule

Solution

Taking five ordinates from X = 0 to X = 1

5 – 1 = 4 number of strips

$$d=\frac{1-0}{4}=0.25$$

$$so, \qquad y=\frac{1}{1+x^2}$$

$$\rightarrow y_o=\frac{1}{1+(0)^2}=1$$

$$\rightarrow y_1=\frac{1}{1+(0.25)}=0.9412$$

$$\rightarrow y_2=\frac{1}{1+(0.5)^2}=0.8$$

$$\rightarrow y_3=\frac{1}{1+(0.75)^2}=0.64$$

$$\rightarrow y_4=\frac{1}{1+1^2}=0.5$$

X	0	0.25	0.5	0.75	1
Y	1	0.9412	0.8	0.64	0.5

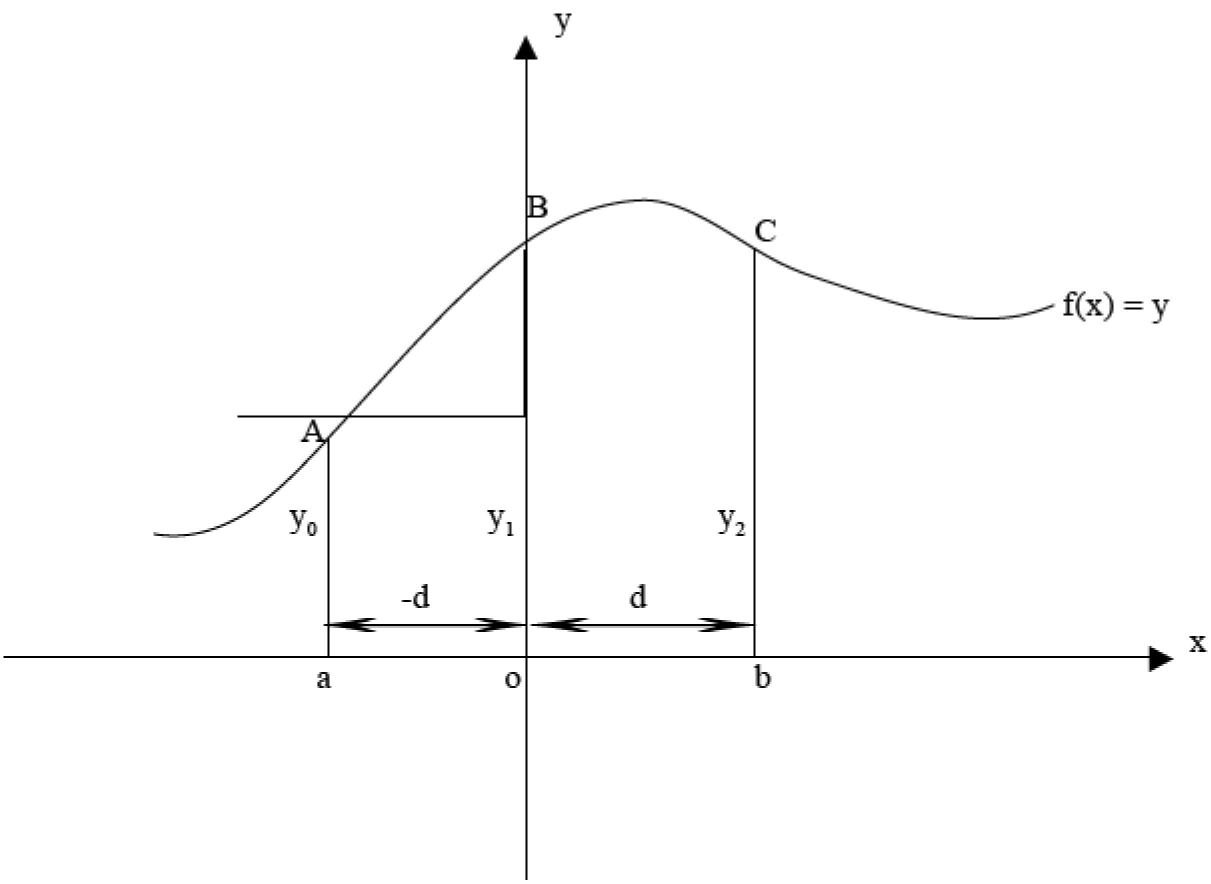
$$\int_0^1 \frac{1}{1+x^2}dx \approx \frac{0.25}{2}\left(1+0.5+2(0.9412+0.64+0.8)\right)$$

$$\approx 0.125\times 6.2624$$

$$\approx 0.7828(to\ 4\ dp)$$

Simpson's rule

Simpson's rule is another method which can be used to find the area under the curve y= f(x) between x = a and x = b



A quadratic equation is fitted (parabola) passing through the three points i.e through A,B,C

Then

$$\int_b^a f(x)dx \approx (ax^2 + bx + c)dx$$

$where\ y = ax^2 + bx + c\ is\ the\ parabola\ through\ the\ ordinates\ y_o, y_1, y_2$

$$\rightarrow y_o = a(-d)^2 + b(-d) + c$$

$$y_o = ad^2 - bd + c \dots \dots \dots .(i)$$

$$y_1 = c \dots \dots \dots .(ii)$$

$y_2 = ad^2 + bd + c \ldots\ldots\ldots (iii)$

$(iii) and\ (i) gives\ y_2 - y_1 = 2bd$

$\rightarrow d = \frac{y_2 - y_1}{2b}$

$(i) + (iii) - 2(ii) gives$

$y_o + y_2 - 2y_1 = 2ad^2$

$\rightarrow a = \frac{1}{2d^2}(y_o + y_2 - 2y_1)$

$area\ of\ first\ strip$

$A_1 = \int_{-d}^{d} yd(x) \approx \int_{-d}^{d} (ax^2 + bx + c)dx$

$\approx \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)_{-d}^{d}$

$\rightarrow A_1 \approx \frac{2ad^3}{3} + 2cd$

$\approx \frac{2d^3}{3} \cdot \frac{1}{2d^2}(y_2 + y_o - 2y_1 d)$

$\approx \frac{d}{3}(y_2 + y_o - 2y_1) + 2y_1 d$

$\approx \frac{d(y_2 + y_o - 2y_1) + 6y_1 d}{3}$

$\approx \frac{d}{3}(y_2 + y_o - 2y_1 + 6y_1)$

$\approx \frac{d}{3}(y_2 + y_o + 4y_1$

$A_1 \approx \frac{d}{3}(y_o + y_2 + 4y_1)$

$similarly \quad A_2 \approx \frac{d}{3}(y_2 + y_4 + 4y_3)$

$A_3 \approx \frac{d}{3}(y_4 + y_6 + 4y_5)$

$i\ general, with\ n\ ordinates\ simpson's\ rule\ is\ total\ area$

$A = A_1 + A_2 + A_3 + \cdots + A_{n-1} + A_n$

$$= \frac{d}{3}(y_o + y_2 + 4y_1) + \frac{d}{3}(y_2 + y_4 + 4y_3) + \cdots + \frac{d}{3}(y_{n-2} + y_n + 4y_{n-1})$$

$$= \frac{d}{3}(y_o + y_2 + 4y_1 + y_2 + y_4 + 4y_3 + y_4 + y_6 + 4y_5 + \cdots + y_{n-2} + y_n + 4y_{n-1})$$

$$total\ area = \frac{d}{3}\left(y_o + y_n + 2(y_2 + y_4 + \cdots) + 4(y_1 + y_3 + y_5 + \cdots)\right)$$

$$\rightarrow \int_a^b f(x)d(x) \approx \frac{d}{3}\left(y_o + y_n + 2(y_2 + y_4 + y_6 + \cdots y_{n-2}) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1})\right)$$

$if\ n\ is\ the\ number\ of\ strips, this\ method\ covers\ an\ even\ number\ of\ strips$

Example

1. $estimate\ to\ 4\ decimal\ places \int_0^1 \frac{1}{1+x^2}dx\ using\ five\ ordinates\ by\ simpson's\ rule$

2. $Use\ simpson's\ rule\ with\ seven\ ordinates\ to\ evaluate \int_0^{0.6} xe^x dx$