PROBABILITY

SUBTOPICS

i) Permutation and combination

ii) Probability of an event

iii) Combined events

iv) Mutual exclusive events

v) Independent events

vi) Application of probability

1. PERMUTATIONS AND COMBINATIONS

Deal with arrangements of objects

Factorial notation

Consider the following pattern

2 x 1 = 2

3 x 2 x 1 = 6

4 x 3 x 2 x 1 = 24

5 x 4 x 3 x 2 x 1 = 120

6 x 5 x 4 x 3 x 2 x 1 = 720

Generally

If you have and different objects, you can get

n x (n – 1) x (n – 2) x (n – 3) x….. 3 x 2 x 1 different arrangements

n x (n – 1) x (n – 2) x (n – 3) x…… x 3 x 2 x 1 is called n – factorial denoted by n!

Examples

2! = 2 x 1= 2

3! = 3 x 2 x 1 = 6

4! = 4 x 3 x 2 x 1 = 24

5! = 5 x 4 x 3 x 2 x 1 = 120

6! = 6 x 4 x 3 x 2 x 1 = 720

0! = 1

Exercise

1. Evaluate each of the following

a) 6!

b) 8! / 3!

c) (7 – 2)!

d) 3! / 0!

e) 5! / (5 – 2)! 2!

2. Simplify

a) $\frac{n!}{(n+1)!}$

b) $\frac{(n+1)!}{(n-1)!}$

c) $\frac{(2n)!}{(2n+2)!}$

3. Write the following in factorial form

a) 4 x 3 x 2 x 1

b) 6 x 5 x 4

c) $\frac{(7 \times 6) \times (3 \times 2 \times 1)}{2}$

Solution

1. a. 6! = 6 x 4 x 3 x 2 x 1 = 720

b. $8!/3! = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{40320}{6} = 6720$

c. (7 – 2) !

5! = 5 x 4 x 3 x 2 x 1 = 120

d. $3!/0! = \frac{3 \times 2 \times 1}{1} = 6$

e. 5! / (5 – 2)! 2!

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1\,(2 \times 1)} = \frac{120}{12} = 10$$

2 a. $$\frac{n!}{(n+1)!} = \frac{n \times (n-1) \times (n-2) \times \ldots\ldots \times 3 \times 2 \times 1}{(n+1) \times n \times n\,(n-1) \times (n-2) \times \ldots\ldots \times 3 \times 2 \times 1}$$

$$= \frac{n!}{n+1 \times n!} = \frac{1}{n+1}$$

$$\frac{(n+1)!}{(n-1)!} = \frac{(n+1) \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1}{(n-1)(n-2) \times n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1}$$

$$= \frac{n+1}{(n-1)(n-2)}$$

$$\frac{(2n)!}{(2n+2)!} = \frac{2n \times (2n-1) \times (2n-2) \times \ldots \times 2 \times 1}{(2n+2)(2n+1) \times 2n(2n-1) \times \ldots \times 2 \times 1}$$

$$= \frac{2n!}{(2n+2) \times (2n+1) \times 2n!}$$

$$= \frac{1}{(2n+2) \times (2n+1) \times 2n!}$$

$$= \frac{1}{(2n+2)(2n+1)}$$

3. a. $4 \times 3 \times 2 \times 1 = 4!$

b. $6 \times 5 \times 4 = \frac{6!}{3!}$

c. $\frac{(7 \times 6) \times (3 \times 2 \times 1)}{2} = \frac{7! \times 3!}{2}$

$$= \frac{(7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)}{(5 \times 4 \times 3 \times 2 \times 1) \times (2 \times 1)} = \frac{7! \times 3!}{5! \times 2!}$$

PERMUTATION AND COMBINATIONS

WAYS OF ARRANGING UNLIKE OBJECTS

Examples

1. How many different three numbers can be formed using the digits 9, 6 and 3

Solution

Digits 9, 6, 3

Ways of science 3 3 3

The total number of ways is 3 x 3 x 3 = 27

2. How many 3 letters groups can be formed using the letters 0, N and W provide each letter is used once?

Solution

Letters	O	N	W
Ways of selective	3	2	1

Total number of ways = 3 x 2 x 1

= 6

PERMUTATIONS

Is arrangement of objects in a particular order.

Example

1. How many ways three letters A B C can be arranged taking three at a time?

Solution

ABC, BCA, BAC, CAB, CBA, ACB

∴ There are 6 ways ie $3p_3 = 6$

Note:

i) The upper 3 indicates the total number of objects

ii) The lower 3 indicates the number of items used in making arrangements

ii How many three letters A, B, C can be arranged taking two at a time?

Solution.

AB, BA, CA, AC, BC, CB

There are 6 ways

i.e. ${}^3p_2 = 6$

Note

The order of arrangement matters

i.e. AB is different permutation from BA

Generally

To find ${}^np_r$ i.e. the number of permutation of n different objects taking r at a time can be arranged as follows.

- The 1st place can be filled in n different ways
- The 2nd place can be filled in (n – 1) different ways
- ‘’ 3rd place ‘’ ‘’ ‘’ ‘’ (n – 2) ‘’ ‘’
- rth ‘’ ‘’ ‘ ‘ (n – r + 1) ‘’ ‘’

${}^nP_r = n(n-1)(n-1) \ldots\ldots (n-r+1)$ ……..i

eqn (i) can be multiplied and divided by (n – r) (n – r – 1) …….x 3 x 2 x 1

eqn i) becomes

$$ {}^np_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1) \times (n-r-1)\ldots\ldots 3 \times 2 \times 1}{(n-r)(n-r-1)\ldots\ldots\ldots\ldots \times 3 \times 2 \times 1} $$

$$ {}^np_r = \frac{n!}{(n-r)!} \ldots\ldots\text{ ii} $$

Examples

1) How many two letters patterns can be formed from the letter six?

Solution

$n = 3, r = 2$

$$ {}^np_r = \frac{n!}{(n-r)!} $$

$$ {}^3p_2 = \frac{3!}{(3-2)} = \frac{3!}{1!} = \frac{6}{1} = 6 $$

2) How many arrangements can be made from the word mathematics?

solution

m = 2, a = 2, t = 2

n =11

$$P = \frac{n!}{m!t!a!}$$

$$p = \frac{n!}{2!2!2!} = \frac{11!}{2!2!2!} = \frac{39,916,800}{8}$$

= <u>4, 989,600</u>

<u>COMBINATIONS</u>

- Is the selection of objects where order is not important

Example (1)

Three letters ABC may be arranged taking three at a time as follows

ABC, BCA, BAC, CAB, CBA, ACB

- These are different permutations but the same combination
- Denoted by, ${}^3C_3 = 1$

Examples (2)

We may take the same letters; find the numbers of selections of the letters ABC, taking two at a time

AB, AC, ~~BC~~, BA, ~~CA~~, ~~CB~~

AB, AC, BC

${}^3C_2$

Examples (3)

In how many ways can r objects be chosen from n unlike objects?

The number of combination of n objects taking at time can be arranged in r!

The number of combination of n objects taking r at a time can be arranged in r

The numbers of permutations

$= r! \times {}^nC_r$

$${}^np_r = r! \times {}^nC_r$$

$${}^nC_r = \frac{{}^np_r}{r!} \ldots\ldots ii$$

But $${}^np_r = \frac{n!}{(n-r)!}$$

$${}^nC_r = \frac{n!}{(n-r)!\,r!} \ldots\ldots (iii)$$

Note:

i) ${}^nC_r$ may be written as $\binom{n}{r}$ or $C(n, r)$

ii) ${}^nC_o = 1$, ${}^nC_n = 1$, ${}^nC_1 = n$

Examples (4)

A mixed hockey team containing 5 men and 6 women is to be chosen from 7 men and 9 women in how many ways can this be done?

Solution

5 men can be selected from 7 men in, ${}^7C_5$ ways

6 women can be selected from 9 men in, ${}^9C_5$ ways

The combination will be,

${}^{7}C_{5} \times {}^{9}C_{6}$

$$= \frac{7!}{(7-5)!5!} \times \frac{9!}{(9-6)!6!}$$

$$= \frac{5040}{240} \times \frac{362,880}{4320}$$

= 21 x 84

= 1,764 ways

Examples (5)

Tabulate the different selections of two letters that can be made from the letters TAKEN, deduce the value of ${}^{5}C_{2}$

Solution

TAKEN

TA, TK, TE, TN, AK, AE, AN, KE, KN, EN

$${}^{5}C_{2} = \frac{5!}{(5-2)!\,2!} = \frac{120}{12} = 10$$

Examples 6

In how many ways can a cricket team be selected from 13 players? Hint a cricket team has 2 players.

Solution

$${}^{13}C_{2} = \frac{13!}{(13-2)!\,2!} = \frac{6,227,020,800}{79,833,600} = 78$$

Examples 7

In ow many ways can a football team of 11 players can be chosen from a class of 15? In how many ways can be the 4 spectators be chosen from the class of 15?

Solution

i) $^{15}C_{11} = \frac{15!}{(15-11)!\,11!} = 1365$

ii) $^{15}C_4 = \frac{15!}{(15-4)!\,4!} = 1365$

Note: $^nC_r = {}^nC_{n-r}$

Exercise

1. From a list of 30 books, how many different groups of 22 books can be selected?

Solution

$^nC_r = \frac{30!}{(30-22)!\,22!}$

$= \underline{5'852'925}$

2. How many different netball teams of 7 members can be formed from 18 players?

Solution

$^nC_r = {}^{18}C_7$

$= \frac{18!}{(18-7)!\,7!}$

$= \underline{31'824}$

3. A circle has 10 marked points, how many different & three – sided figures which can be formed by joining any three of these points

Solution

$$^{10}C_3 = \frac{10!}{(10-3)!\,3!}$$

$$= \underline{120}$$

4. How many different committees comprising of 20 people can be formed from 25 people?

Solution

$$^{25}C_{20} = \frac{25!}{(25-20)!\,20!} = \underline{53'130}$$

5. If a plane paper has 9 points in which no three points appear on the same straight line, how many distinct triangles can be formed by joining any three points?

Solution

$$^{9}C_3 = \frac{9!}{(9-3)!\,3!} = \underline{84}$$

PROBABILITY OF AN EVENT.

In any experiment, depending on the number of trials.

Sample space, S
Is the set of all possible outcomes.

Example;
Tossing of dice once;
S = {1, 2, 3, 4, 5, 6}.

Tossing of coin once.

S = {H, T}.

Event, E
- Is the specified outcomes.

Example;
Tossing of a dice once the specified outcome is odd number.
Sample space, S = {1, 2, 3, 4, 5, 6}
Event, E = {1, 3, 5}

Possibility
- Is the occurrence of any expected event.

Note:
Sample space and event are usually denoted by capital letters like A, B and E.

Definition.
Possibility of an event is the ratio of number of events to the number of sample space.

$$P(E) = \frac{n(E) = \text{number of events.}}{n(S) = \text{number of sample space.}}$$

Example;
A coin is tossed twice, what is the probability of getting two tails?

Solution
By using probability table.

		First Toss	
		H	T
Second Toss	H	HH	HT
	T	TH	TT

Sample space, S = {HH, HT, TH,TT}
Event, E = {TT}
Thus,
n(S) = 4, n(E) = 1

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$$

Exercise

1. Write the possibility set for each of the following experiments;
   a) Expectation of the football match between two teams
   b) A die is flipped and a face showing up is recorded.

2. A student is taken at random from a class of 15 boys and 10 girls. What is the probability that the student taken at random is not a girl?

3. A number between 19 and 31 inclusive is chosen at random. What is the that;
   a) The number chosen is odd?
   b) The number chosen is prime?

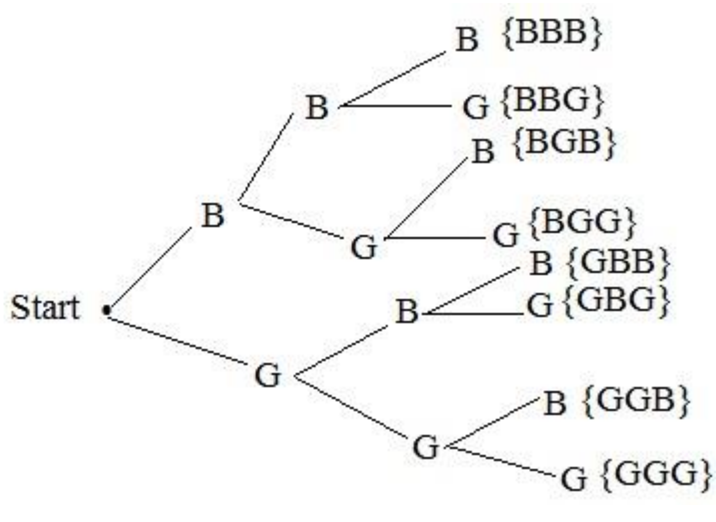
4. Taking the sample space for the total number of all possible outcomes when a pair of dice is flipped, find on a single flip of a pair of dice, the probability of obtaining;
   a) A sum of ten.
   b) A odd sum.
   c) A sum less than fire.

COMBINED EVENTS.

Combined events are those events that can be represented by two or more simple events.
The events/outcome can easily be obtained by using a tree diagram or a table.

Example;
Use a tree diagram to list the sample space showing the possible arrangements of boys and girls in a family of exactly three children. What is the probability that,
a) All children are girls.
b) Two children are girls and one is a boy.
c) At least one of the children is a boy.
d) None of the children are girls.

Solution
The tree diagram.



The outcome can be summarized as;

S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

a) E_1 = {GGG}; n(S) = 8

$n(E_1) = 1$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1}{8}$$

b) E_2 = {BGG, GBG, GGB)

$n(E_2) = 3$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{8}$$

c) E_3 = {BBB, BBG, GBB, BGB, BGG, GBG, GGB}

$n(E_3) = 7$

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{7}{8}$$

d) E_4 = {BBB}

$n(E_4) = 1$

$$P(E_4) = \frac{n(E_4)}{n(S)} = \frac{1}{8}$$

Example 2:

In a single toss of a pair of dice, find the probability of obtaining a sum of

i) 9
ii) 12
iii) 9 or 12

Solution.

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

n(S) = 6 x 6 = 36

Let E_1 be the event of a sum of 9
E_2 be the event of a sum of 12

a) Then, $E_1 = \{(3,6), (4,5), (5,4), (6,3)\}$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

b) Also, $E_2 = \{(6,6)\}$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{1}{36}$$

c) The sum is 9 or 12.

$P(E_1 + E_2) = P(E_1) + P(E_2)$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)}$$

$$= \frac{1}{9} + \frac{1}{36}$$

$$P(E_1 \text{ or } E_2) = \frac{5}{36}$$

Exercise

1. A fraction is written by selecting the numerator from the digit 1,2 and 4 and the denominator from the digits 5, 6 and 7. Find the probability that the fraction written will be;
   a) Less than two-third
   b) Less or equal to a half

2. Two coins and a die are simultaneously tossed. What is the probability that the number less than three, a head and a tail will show up.

3. Two dice are flipped. Find the probability of getting
   a) At least one 5.
   b) A total score of 6.
   c) A total score not divisible by 3.

4. Three coins are tossed simultaneously. Find the probability that
   a) Three tails appear.
   b) At least two tails appear.
   c) Two heads and one tail appear.

Note:

Probabilities are expressed as fractions or percentages and obey the following rules.

i) Rule of range.

Let E be an event, then $0 \leq P(E) \leq 1$

i.e Probability of an event lies between 0 and 1.

If P(E) = 0, then E can not occur.

If P(E) = 1, then E is certain to occur.

ii) Rule of complement.

If E is an event, then

$P(E) + P(E)' = 1$

Where, $P(E)' = 1 - P(E)$, Where $P(E)'$ is the probability of an event not occurring.

MUTUALLY AND NON-MUTUALLY EXCLUSIVE EVENTS

MUTUALLY EXCLUSIVE EVENTS.

- These are events that can not occur concurrently.
- Occurrence of one event precludes the occurrence of the other event.

For mutually exclusive events, P(AnB) = 0

$$\boxed{P(AuB) = P(A) + P(B)}$$

NON-MUTUALLY EXCLUSIVE EVENTS.

- These are events that can occur concurrently.
- The occurrence of one will not hinder the occurrence of the other. They may both occur simultaneously.

For Non-mutually exclusive events,

$$\boxed{P(AuB) = P(A) + P(B) - P(AnB)}$$

Example 1;

A boy contains 3 red balls, 4 blue balls, 5 white balls and 6 green balls. Which are identical. A ball is drawn at random, Find the probability that it is either;

a) red or blue.
b) red or white.
c) blue or white or green.
d) red or blue or green or white.

Solution.

$$n(S) = 3 + 4 + 5 + 6 = 18$$

$$P(r) = \frac{3}{18},\ P(b) = \frac{4}{18},\ P(w) = \frac{5}{18},\ P(g) = \frac{6}{18}$$

a) $P(r \text{ or } b) = P(r) + P(b)$

$$= \frac{n(r)}{n(s)} + \frac{n(b)}{n(s)}$$

$$= \frac{3}{18} + \frac{4}{18}$$

$$P(r \text{ or } b) = \frac{7}{18}$$

b)

$$P\text{ (r or w)} = P(r) + P(w)$$
$$= \frac{3}{18} + \frac{5}{18}$$
$$P\text{ (r or w)} = \frac{8}{18}$$

c)

$$P(\text{b or w or g}) = \frac{4}{18} + \frac{5}{18} + \frac{6}{18}$$
$$P(\text{b or w or g}) = \frac{15}{18} = \frac{5}{6}$$

Example 2;
A card is chosen at random from a standard deck of 52 playing cards. What is the probability of getting a king or a club?

Solution
The two events are non-mutually exclusive events.

P(King or Club) = P(King) + P(Club) - P(King and Clubs)

$$P(\text{King or Club}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$
$$P(\text{King or Club}) = \frac{16}{52} = \frac{4}{13}$$

Exercise.
1. If P(A) = 0.1, P(B) = 0.9 and P(AnB) = 0.2. Find P(AuB) given that A and B are non-mutually exclusive events

2. If P(A) = 0.2 and P(B) = 0.5, Find P(AuB) given that A and B are independent events.

3. If P(A) = 1/3 and P(B) = I/2, Find P(AuB) given that A and B are mutually exclusive events.

4. Find the probability of drawing an Ace or a King in a single draw from a deck of 52 playing cards.

5. A fair die is tossed once. Find the probability of getting a number greater than three or an even number.

INDEPENDENCE AND DEPENDENT EVENTS.

INDEPENDENT EVENTS.

- Are those events in which the occurrence of one event has nothing to do with the occurrence or non occurrence of the other event.

Two events A and B are said to be independent if P(AnB) = P(A) x P(B). This is called multiplicative rule for independent events.

Example.

An electronic device has two independent components with reliability of 0.82 each. The device work only if both components are functional, What is the probability that the device will not work?

Solution

Let A be the event of first component

B be the event of second component

P(AnB) = P9A) x P(B)

= 0.82 x 0.82

P(AnB) = 0.6724

Therefore,

The probability that the device will not work.

P(AnB)' = 1 - P(AnB)

= 1 - 0.6724

P(AnB)' = 0.3276

Exercise

A bag contains 3black marbles and 2 white marbles.

a) A marble is taken at random from the bag and then replaced. A second marble is chosen. What is the probability that;

i) They are both black?

ii) One is black and the other is white?

b) Find the probability if the two marbles are chosen without any replacement.

DEPENDENT EVENTS.

- Are those events in which the occurance of one event affects the occurance of the other event.

- Two events A and B are said to be dependent if,

P(AnB) = P(A/B) x P(B) or

P(AnB) = P(B/A) x P(A) Where

P(A/B) or P(B/A) is conditional probability.

- It read as probability of A is given that B has occurred/ is certain to occur.

Example 1

If A and B are dependent events such that P(B) = 2/3 and P(AnB) = 1/4. Find P(A/B)

Solution

$$P(AnB) = P(A/B) \times P(B)$$

$$P(A/B) = \frac{P(AnB)}{P(B)}$$

$$= \frac{1}{4} \div \frac{2}{3}$$

$$= \frac{1}{4} \times \frac{3}{2}$$

$$P(A/B) = 3/8$$

Example 2
If A and B are dependent events such that P(B/A) = 3/4 and P(A) = 1/2, Find P(AnB).

Solution.

$$P(AnB) = P(B/A) \times P(A)$$

$$= \frac{3}{4} \times \frac{1}{2}$$

$$P(AnB) = \frac{3}{8}$$

Exercise
1. A coin is tossed and a spinner numbered 1 to 7 is made to spine. Find the probability of obtaining a tail on the coin and an odd number on the spinner.

2. The names of 5 boys and 3 girls were put in a box. One name is picked at random from the box, without replacing the first name, second name is picked at random. Find the probability that both are names of boys.

3. On an interview, four out of ten interviewees got an A grade. If there interviewees are chosen at random without replacement, find the probability that all three got an A on the interview.

APPLICATION OF PROBABILITY

The applications of probability includes;

i) Budget estimates.

ii) Assessing the amount of harvest in agriculture.

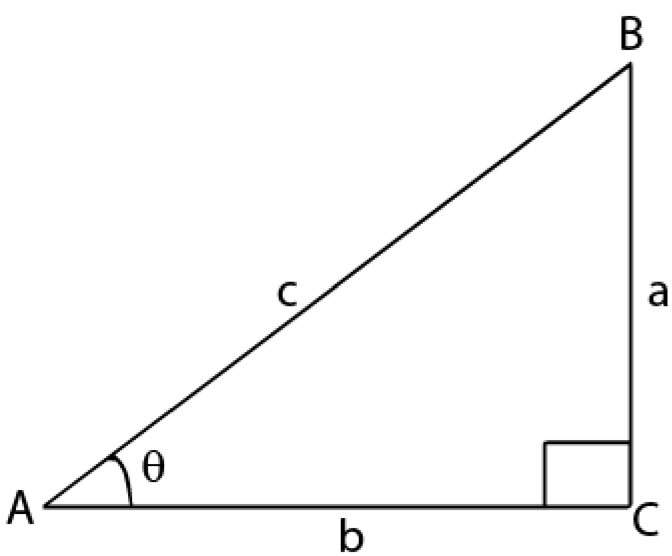
iii) Lottery games.

iv) Human reproductions to plan a baby boy or girl.

v) Risk control in many aspects in our day to day life.

TRIGONOMETRY

Trigonometry deals with the measurements of triangles and problems involving triangles. The trigonometric functions are usually expressed using the various ratios of the side of right angle. Consider figure 8.1



Sine: $\sin\theta = \frac{\text{opposite side}}{\text{Hypotenuse side}} = \frac{a}{c}$

Cosine: $\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse side}} = \frac{b}{c}$

Sin θ = $\frac{a}{c} \div \frac{b}{c} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b}$

Sin θ = tan θ. When cos θ ≠ 0
So, tan θ = $\frac{a}{b}$

Other trigonometric ratios

The reciprocals of sine, cosine and tangent are also trigonometrically ratios. The reciprocal of sine is called the cosecant, the reciprocal of cosine is secant while the reciprocal of tangent is called cotangent.

The short form of cosecant, secant and cotangent of an angle θ are cosec θ, sec θ and cot θ respectively. Thus,

a) $\frac{1}{\tan\Theta} = \frac{\cos\Theta}{\sin\Theta}$, where $\sin\Theta \neq 0$

b) $\frac{1}{\cos\Theta} = \sec\Theta$, where $\cos\Theta \neq 0$

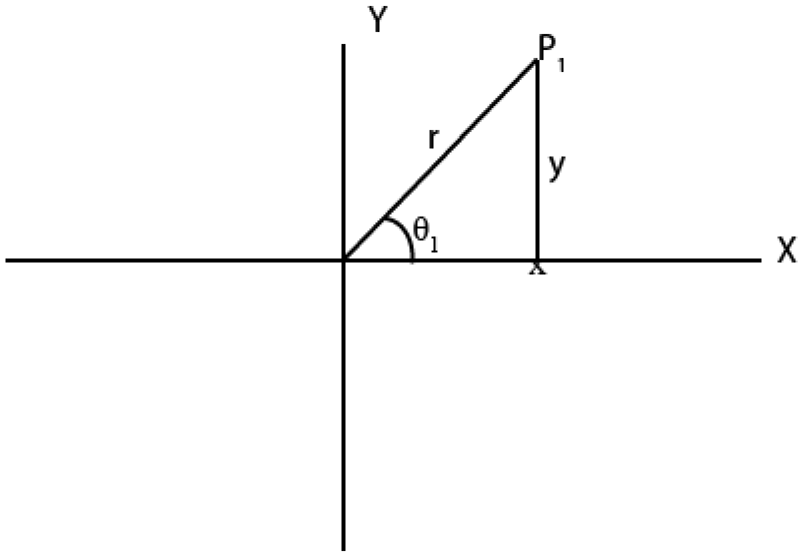
c) $\frac{1}{\sin\Theta} = \text{cosec}\,\Theta$, where $\sin\Theta \neq 0$

Angles of any magnitude

The trigonometric ratios can be studied better if we look at where these angles appear in the quadrant. Angles in the range 0^o to 90^o are in the first quadrant, $90^o – 180^o$, $180^0 – 270^o$ and $270^o – 360^o$ are in the second, third and fourth quadrants respectively.

<u>Angles $0^o – 90^o$</u>

The trigonometric ratios of angles θ, in the first quadrant are all positive information under figure 8.2



Sin $\theta_1 = \frac{y}{r}$ = +ve

Cos $\theta_1 = \frac{x}{r}$ = +ve

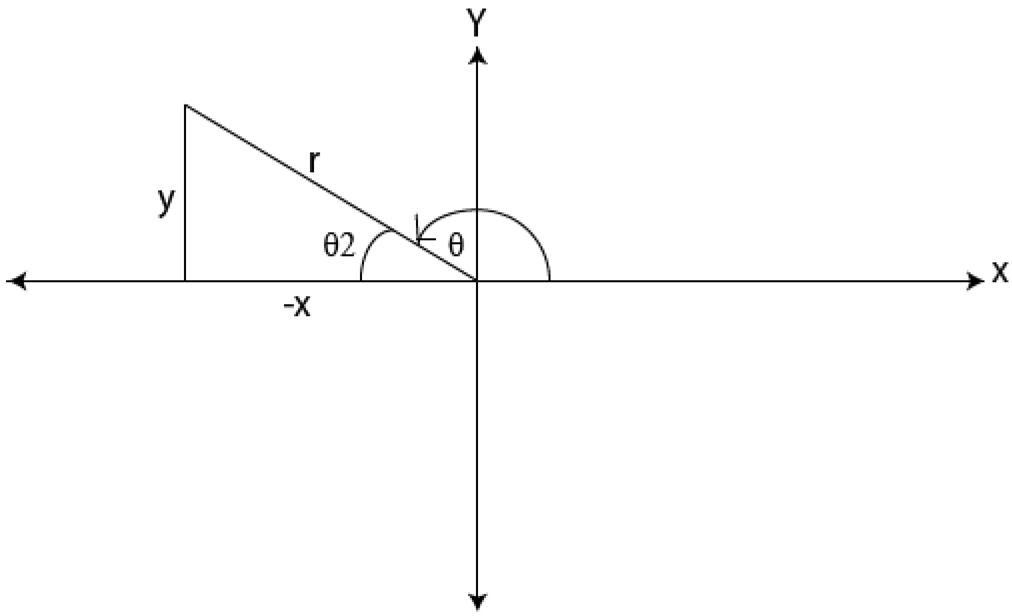
Tan $\theta_1 = \frac{y}{r}$ = + ve

Note

- All angles between 0^o and 90^o are called acute angles
- The size of the angles used to read through mathematical tables are all the time acute angles relative to the horizontal x – axis, ie the angles $\leq 90^o$

Angles 90^o – 180^o

The angles θ in the second quadrant have the same trigonometric ratios as the angle θ_2 i.e $\theta_2 = 180^o - \theta$. In this quadrant, only the trigonometric ratios of sine are positive. See figure 8.3 and the information under it.



$\theta_2 = 180^o - \theta$

Sin θ_2 = sin (180 - θ) = $\frac{y}{r}$ = +ve

Cos θ_2 = cos (180 - θ) = -$\frac{y}{r}$ = -ve

Tan θ_2 = tan (180 - θ) = $\frac{y}{-x}$ = -ve

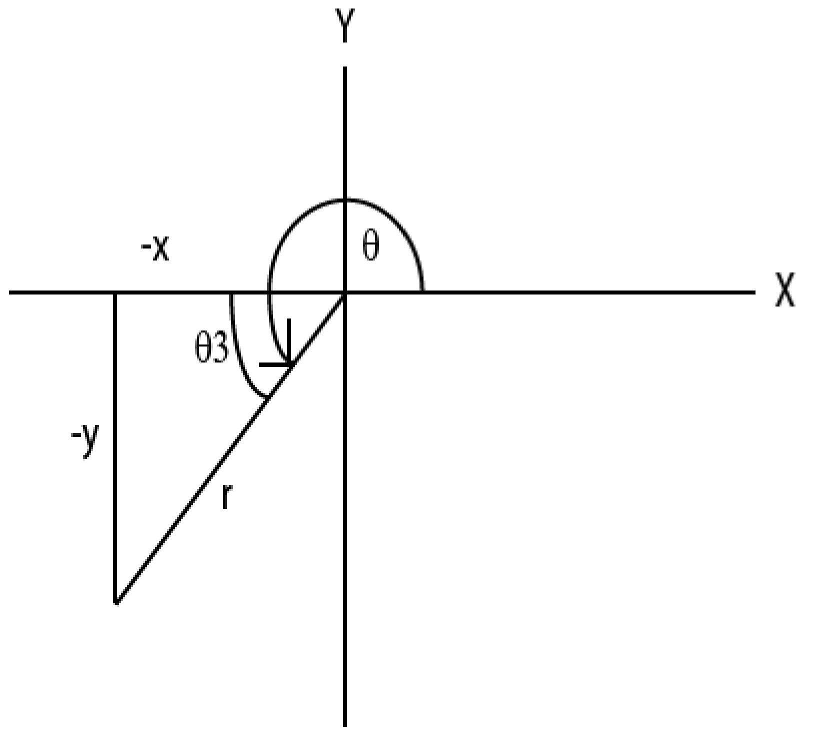
Sin θ_2 is positive but cos θ_2 and tan θ_2 are negative

Note: All angles between 90^o and 180^o are called obtuse angles.

<u>Angles 180^o – 270^o</u>

The angles θ in the third quadrant have the same trigonometric ratios as the angles θ_3. In this quadrant, only the tangents are positive.

See figure 8.4 and the information under it.



$\theta_3 = \theta - 180^o$

Sin θ_3 = sin (θ - 180^o) = -$\frac{y}{r}$ = -ve

Cos θ_3 = cos (θ - 180o) = -$\frac{x}{y}$ = -ve

Tan θ_3 = tan (θ - 180^o) = $\frac{-x}{-y}$ = +ve

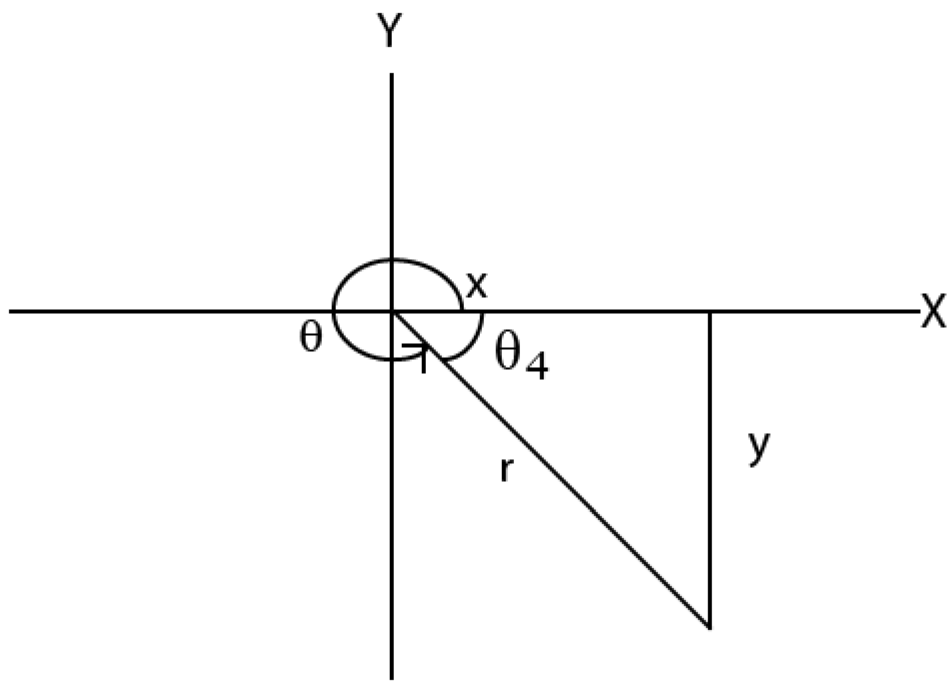
Then, Tan θ_3 is positive but sin θ_3 and cos θ_3 are negative

Note

All angles between 180^0 – 270^0 are called angles of reflection

Angles 270^o – 360^o

The angles θ in the fourth quadrant have the same trigonometric ratios as the angles θ_4. i.e θ_4 = 360^o - θ. In this quadrant only the cosine are positive. See figure 8.5 and the information under it.



$θ_4 = 360 - θ$

Sin $θ_4$ = sin (360 - θ) = $\frac{-y}{r}$ = -ve

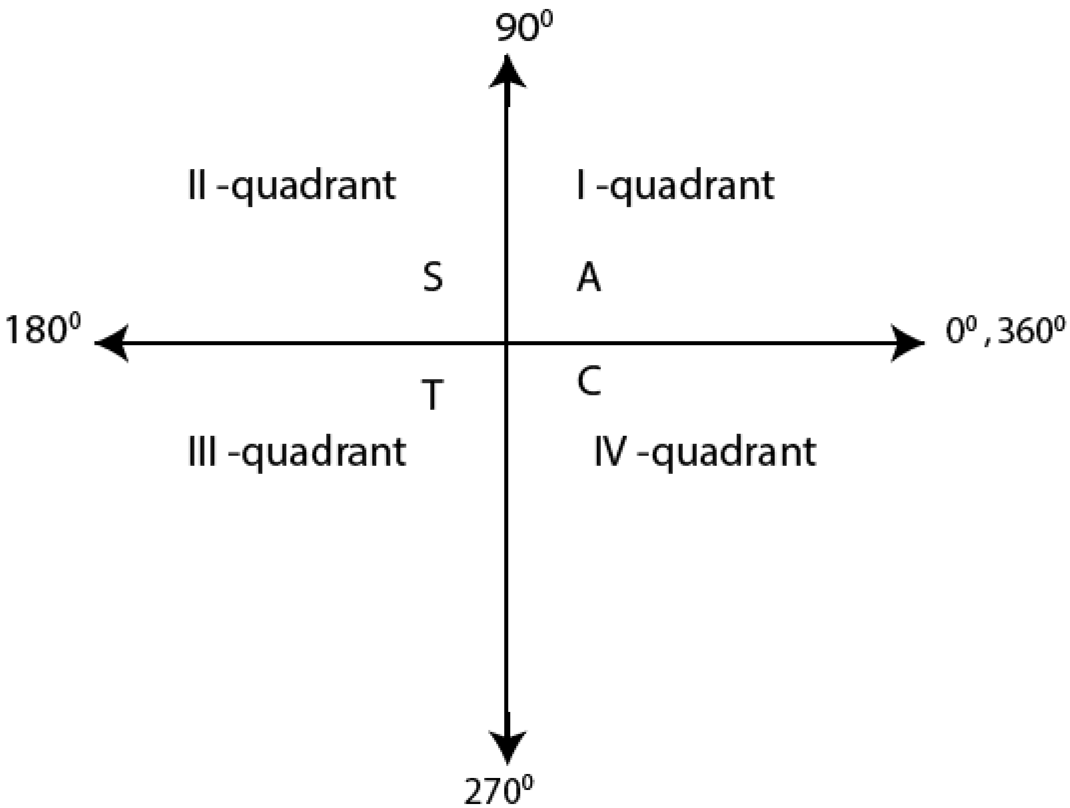
Cos $θ_4$ = cos (360 - θ) = $\frac{x}{r}$ = +ve

Tan $θ_4$ = tan (360 - θ) = $\frac{-y}{x}$ = -ve

Sin $θ_4$ and tan $θ_4$ are all negative but cos $θ_4$ is positive

Note: All angles between 270° – 360° are called angles of reflection.

Generally, the values of trigonometric ratios and their sign are shown in figure 8.6 A stand for all sine, cosine and tangent are positive, S stands for only sine is positive stands for only tangent is positive and C stands for only cosine is positive



Examples 1.1

Evaluate each of the following

i) sin 135^o

ii) Cos 230^o

iii) Tan 315^o

iv) Tan 540^o

v) Cos (920^O)

Solution

i) Sin 135^o = +sin (180^o – 135^o) = +sin 45^o = $\frac{\sqrt{2}}{2}$

ii) Cos 230° = -cos (230° – 180°) = cos 50° = 0. 6428

iii) Tan 315° = -tan (360° – 315°) = tan 45° = -1

iv) Tan 540° = tan (540° – 360°) = tan 180° = 0

v) Cos (920°) = -cos (920° – 720°) = -cos 200°

= -cos (200° – 180°)

= -cos 20°

= -0.40808206

Special Angles

0°, 30°, 60°, 45°, 90°, 180°, 270°, 360°

Consider an equilateral triangle whose side is 2 units

Length BD, Consider △ BDC

By, pathagorous theorem.

$a^2 + b^2 = c^2$

$BD^2 + DC^2 = BC^2$

$BD^2 + 1^2 = 2^2$

$BD^2 = 2^2 - 1^2$

$BD^2 = 3$

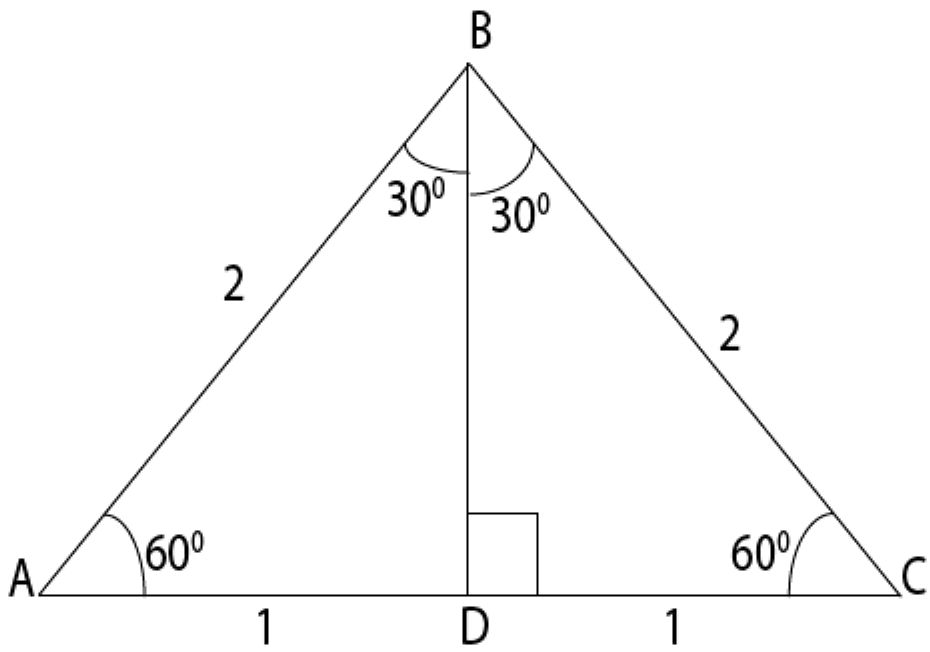
$BD = \sqrt{3}$

Then, sin 60° = $\frac{\sqrt{3}}{2}$ and sin 30° = $\frac{1}{2}$

Cos 60° = $\frac{1}{2}$ cos 30° = $\frac{\sqrt{3}}{2}$

Tan 60° = $\sqrt{3}$ tan 30° = $\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

-



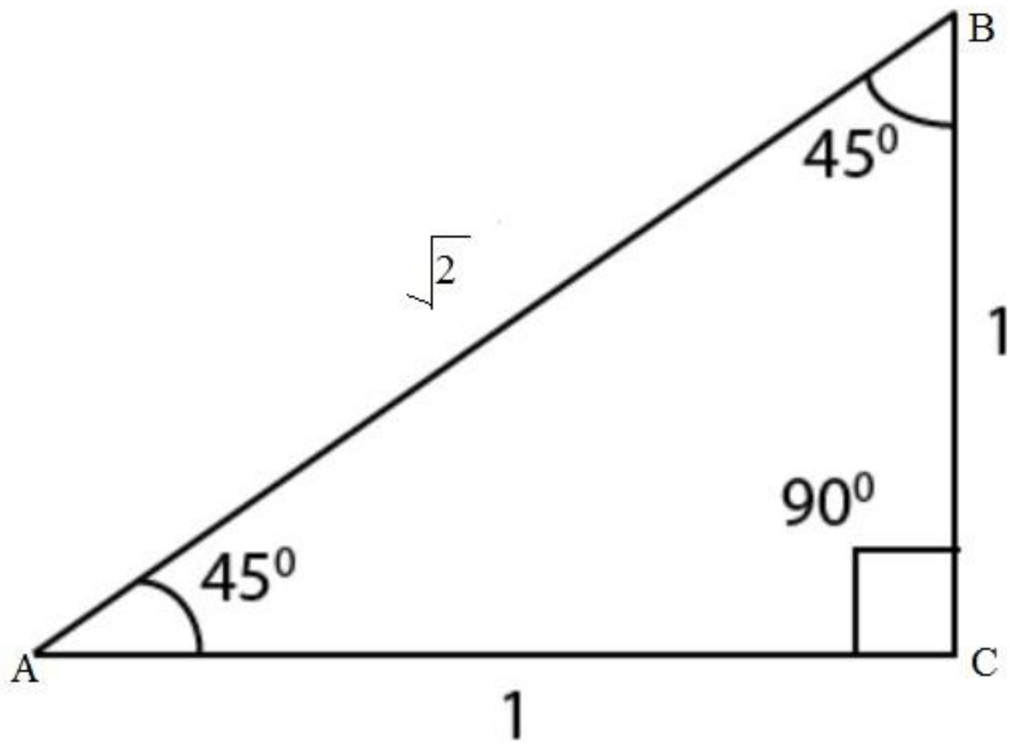
Consider an isosceles triangle ABC below

Similarly,

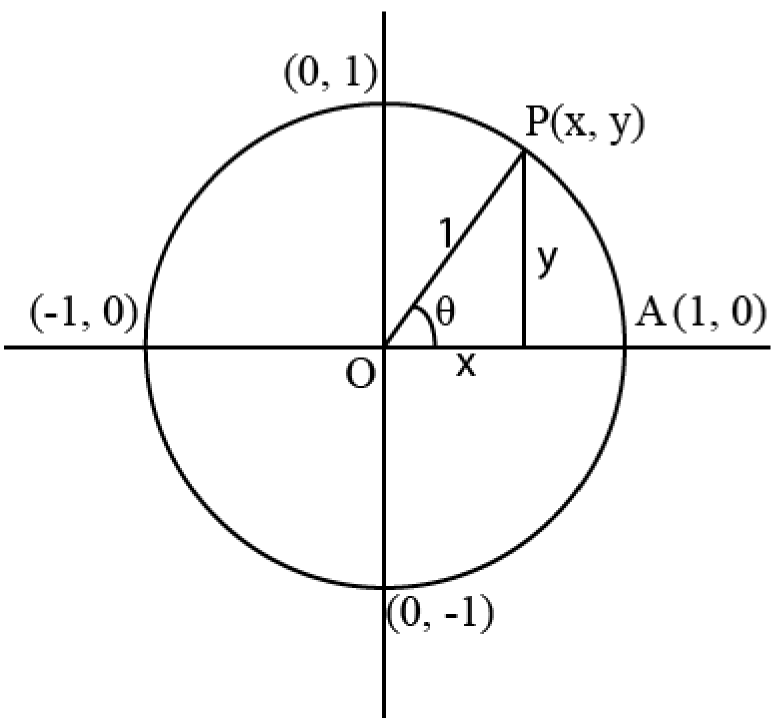
Sin 45° = $\frac{1}{\sqrt{2}}$ x $\frac{\sqrt{2}}{\sqrt{2}}$ = $\frac{\sqrt{2}}{2}$

Cos 45° = $\frac{1}{\sqrt{2}}$ x $\frac{\sqrt{2}}{\sqrt{2}}$ = $\frac{\sqrt{2}}{2}$

Tan 45° = $\frac{1}{1}$ = 1



Consider unit circle



From the diagram you will find that;

i) Cos θ = $\frac{x}{1}$

Cos θ = x

ii) Sin θ = $\frac{y}{1}$

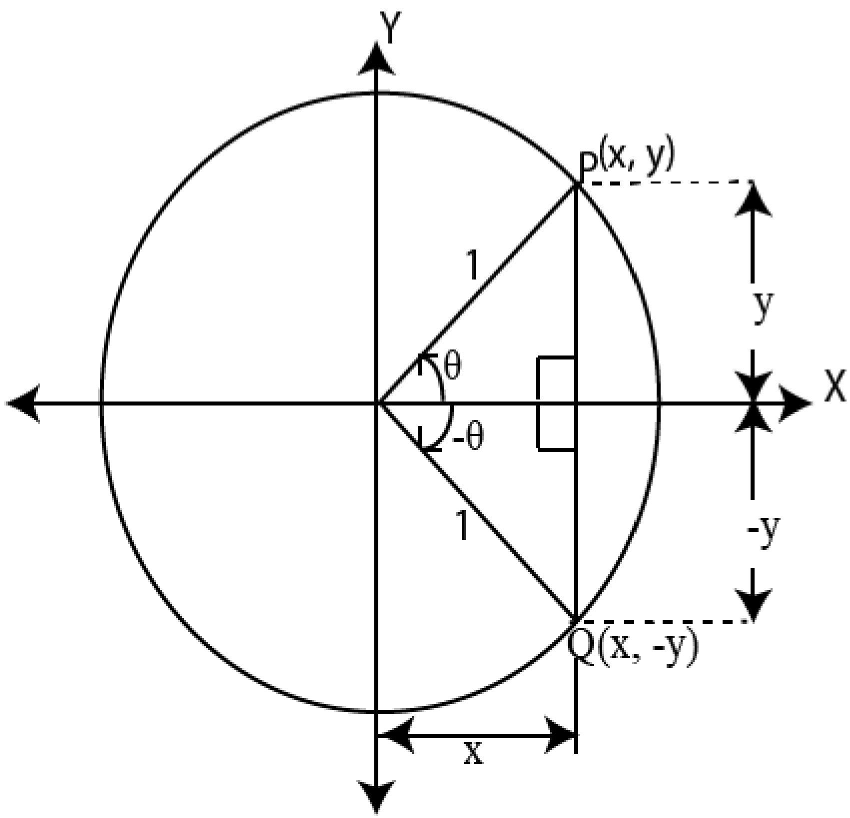
Sin θ = y

If p (x, y) = p (cos θ, sin θ), then

Sin 0^o = 0	sin 90^o = 1	sin 180^o = 0
Cos 0^o = 1	cos 90^o = 0	cos 180^o = -1
Tan 0^o = 0	tan 90^o = ∞ undefined	tan 180^o = 0
Sin 270^o = 0	sin 360^o = 0	
Cos 270^o = 0	cos 360^o = 1	
Tan 270^o = ∞ (undefined)	tan 360^o = 0	

Negative angels

Consider a unit circle in figure 8.10



Let θ be any angle then

Cos θ = $\frac{x}{1}$ = x

Sin θ = $\frac{y}{1}$ = y

a) Sin (-θ) = $\frac{-y}{1}$ = -y

But y = sin θ

-y = -sin θ

Sin (-θ) = -sin θ

b) Cos (-θ) = $\frac{x}{1}$ = x

But x = cos θ

Cos (-θ) = cos θ

c) $\text{Tan}(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$

∴ Tan (-θ) = - tan θ

Example 1.2

Evaluate sin θ and cos θ for the given θ

a) -45^o b) -180^o c) -90^o d) -240^o

Solution

a) Sin (-45^o) = -sin 45^o = $\frac{\sqrt{2}}{2}$

Cos (-45^o) = cos 45^o = $\frac{\sqrt{2}}{2}$

b) Sin (-180^o) = -sin (180^o) = 0

Cos (-180o) = -sin 180^o = 1

c) Sin (-90^o) = -sin 90^o = -1

Cos (-90^o) = cos 90 = 0

d) Sin (-240^o) = -sin (240^o)

= -sin $(240^o – 180^o)$

= sin 60^o

= + $\frac{\sqrt{3}}{2}$

Cos (240^o) = cos (240^o) = cos (240^o)

$= -\cos (240^o - 180^o)$

$= {}^-½$

RADIANS

Definition

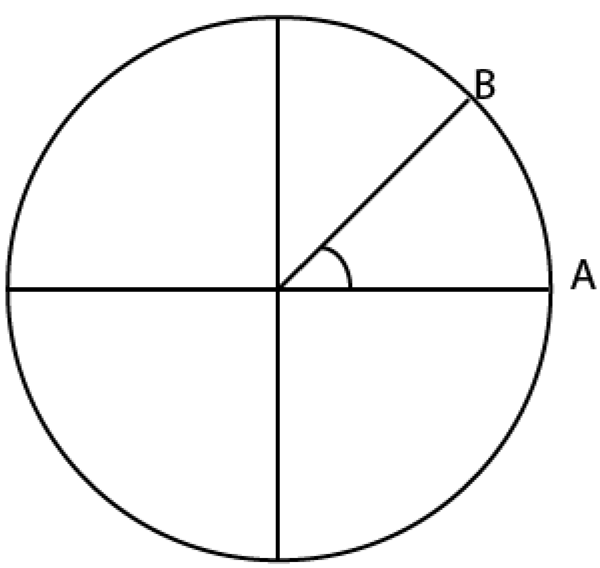
1 radian is an angle subtended at the centre of the circle of the circle by an are equal in length to that radius.

i.e

$$\frac{\theta}{360^\circ} = \frac{l}{circ} = \frac{l}{2\pi r}$$

But θ = 1 radian

Circumference = 2 πr



$l = r$

$\frac{\theta}{360^0} = \frac{\ell}{2\prod r}$

$\frac{1\,radian}{360^0} = \frac{r}{2\prod r} = \frac{1\,radian}{360^0} = \frac{r}{2\prod r}$

$\frac{1\,radian}{180^0} = \frac{1}{\pi}$

π Radian = 180^o

Note: It is customary to omit the unit radian

$\pi = 180^o$

Exercise 1.3

1. Convert the following into degrees

   a) $\frac{\pi}{3}$ b) $\frac{1}{4\pi}$ c) $\frac{3}{2\pi}$ d) $\frac{19}{6}\pi$

   e) 5 f) $\frac{\pi}{10}$ g) $\frac{4}{3\pi}$ h) $\frac{2}{3\pi}$

   j) $\frac{-\prod}{3}$

2. Convert the following into radians

   a) 120^o b) 4^o c) -90^o d) 15^o

   e) 60^o f) 240^o g) -315 h) -90^o

   i) 357^o j) 300^o k) -630^o

3. Solve for x such that $0^o \leq x \geq 360^o$

a) Cos x = -1

b) Cos x = $-\frac{1}{2}$

c) Sin x = $-\frac{1}{2}$

d) Sin x = 0

e) Cos x = $\frac{1}{2}$

f) Sin x = $\frac{1}{2}$

g) Tan x = undefined

h) Tan x = 0

i) Tan x = -1

j) Tan x = 1

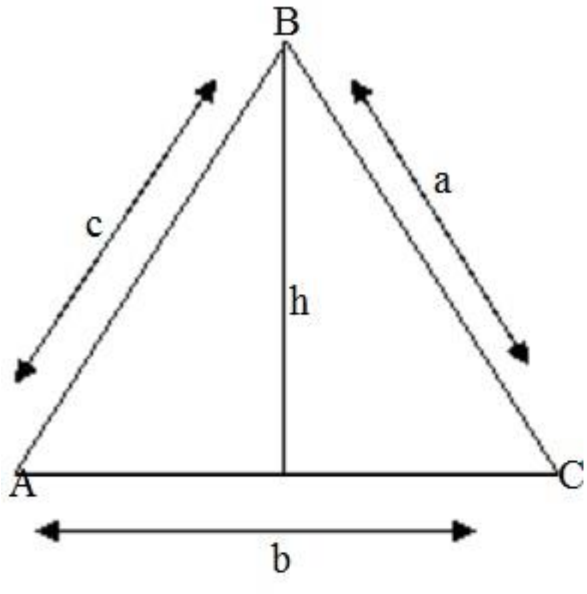
4. Evaluate sin θ if the terminal side of θ contains the given point

a) P (10, -4) b) P (3, 4) c) P (3, -5) d) P (-1, 3)

e) P (-8, -13)

5. If cos θ = -2/5 and sin θ < 0, find sin θ

6. If cos θ = 5/3 and sin θ < 0, find sin θ.



→ Sin A = h/c h=c sin A

→ Sin C = h/a h=a sin C

Sin A / a = sin C / c

Similarly, by drawing the altitude from A to BC.

b sin C = c sin B

Sin C / c= sin B / b

Hence; sin A /a = sin B / b = sin C / c

It is called the sine rule.

Note:

a) When two angles and one side of any triangle are known, the sine rule can be used to solve the triangle

b) When two sides and an angle opposite to any of the two given sides are known, the sine rule can be used to solve the triangle.

Example

7. In triangle ABC, B = 39^0, C = 82^0, a = 6.73 cm find C

Solution

From $\frac{\sin c}{c}$ $\frac{\sin A}{a}$

$= \frac{\sin 82^0}{c} = \frac{\sin A}{6.73\ cm}$

But A = 180° – (39° + 82°)

= 59°

$= \frac{\sin 82^0}{c} = \frac{\sin 59^0}{6.73\ cm}$

C = 6.73cm x $\frac{\sin 82^0}{\sin 59^0}$ = 7.78 cm

Example 1.4

Find the remaining angles of the AABC in which a = 12.5 cm , c = 17.7 cm and C = 116^0

Solution

$\frac{\sin A}{a} = \frac{\sin C}{c} = \frac{\sin A}{12.5} = \frac{\sin 116^0}{17.7}$

= sin A = $\frac{\sin 110^0\ x\ 125}{17{,}7}$

A = $\sin^{-1}(0.6347)$

A = $39^0.24$

B = 180° – 155 = B = $\underline{24^{o}\ 36}$

Exercise 1.5

1. Solve completely the triangles in which

   a) b = 6m, A= 80°, C = 46°

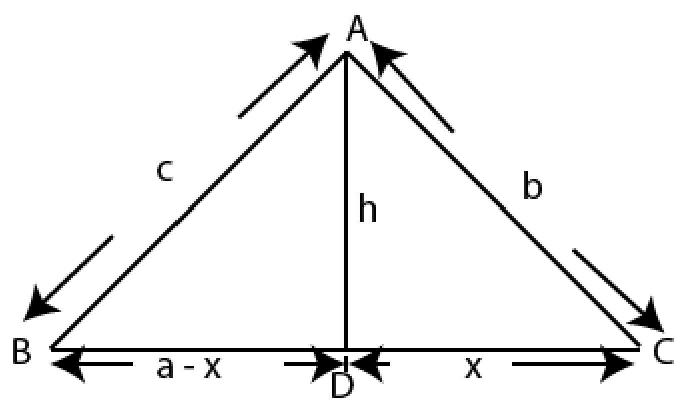
   b) A = 50°, B = 60°, a = 4m

   c) a = 400m, A = 31°20, B = 70°40

   d) A = 115°, a = 65m, b = 32m

Cosine rule

Consider the triangle ABC



By using Pythagoras theorem

$C^2 = (x - a)^{2} + h^2$

$C^2 = x2 - 2ax + a^2 + h^2$

$C^2 = x^2 + h^2 + a^2$

$C^2 = a^2 + b^2 - 2ax$

But $\frac{x}{b}$ = cosC

$C^2 = a^2 + b^2 - 2ab \text{ Cos } C$

For angle B

$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$

For angle A

$a^2 = b^2 + c^2 - 2bc \text{ Cos } A$

Note

a) When two sides and an angle between the two sides are included then the angle between the two lines of the cosine rule can be used to solve the triangle by using the cosine rule.

b) We can also solve the triangle by cosine rule if all the three sides are given

Example 1.6

Given a triangle with a = 50cm, c =60cm, and B = 100^o .Find b

Solution

$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$

$= (50)^2 + (60)^2 - 2(50)(60) \text{ Cos } 100^o$

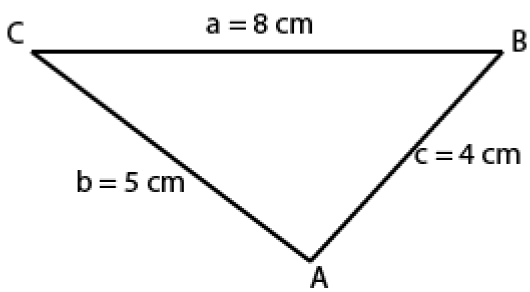
$= 3600 + 2500 - 6000 (-0.1736)$

$= \underline{7141.6 \text{ cm}}$

Example 1.7

Given a = 8cm, b = 5cm, and c = 4cm, find the smallest and largest angles.

Solution



i) To find A we use

$a^2 = b^2 + c^2 – 2bc \text{ Cos } A$

$64 = 25 + 16 – 40 \text{ Cos } A$

$40\cos A = 41 – 64$

$40\cos A = -23$

$\text{Cos } A = \frac{-23}{40} = -0.5750$

$A = 125^o10$

ii) To find C we use

$c^2 = a^2 + b^2 – 2ab\text{Cos } C$

$16 = 8^2 + 25 – 2 (8) (5) \text{ Cos } C$

$16 = 64 + 25 - 2(8)(5)\text{Cos } C$

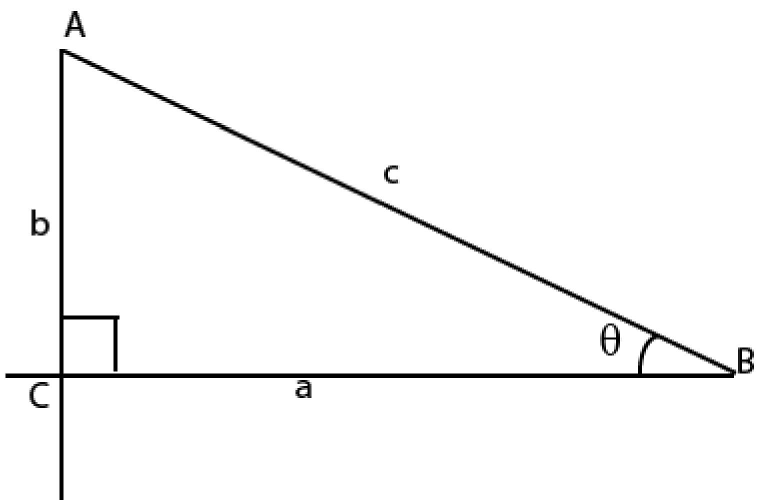
$80\cos C = 89 – 16$

$\text{Cos } C = \frac{73}{80} = 0.9125$

$C = 24^o10$

Trigonometric identities

Figure 8.15 show the right angled triangle with sides a, b, c and angles A, B, C



By the use of Pythagoras theorem

$a^2 + b^2 = c^2$

Divide throughout by c^2

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \left(\frac{c}{c}\right)^2$$

But $\frac{a}{c}$ = cos θ, $\frac{b}{c}$ = sin θ

$\cos^2\theta + \sin^2\theta = 1$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \text{(divide by } \cos^2\theta \text{ on both sides)}$$

$$1 + \tan^2\Theta = \sec^2\Theta$$

$$\frac{\cos^2\Theta}{\sin^2\Theta} + \frac{\sin^2\Theta}{\sin^2\Theta} = \frac{1}{\sin^2\Theta} \quad \text{(dividing by } \sin^2\Theta \text{ on both sides).}$$

$$\cot^2\Theta + 1 = \operatorname{cosec}^2\Theta$$

Exercise 1.8

1. Use the cosine rule to solve the following triangles

a) $A = 60^o$, $b = 3cm$, $c = 4cm$

b) $A = 100^o$, $b = 8cm$, $c = 10cm$

c) $A = 50^o$, $b = 5cm$, $c = 8cm$

d) $a = 9cm$, $b = 10$, $C = 120^o$

e) $a = 5$, $b = 10$, $c = 120^0$

f) $a = 10cm$, $b = 20cm$, $c = 120cm$

2. Find the largest and smallest angles of triangle whose side area a= 5cm b = 10cm and c = 12cm

3. Two people are on opposite sides of a hill and are 600m apart. If the angles of elevation from these people to the top of the hill are 19^o and 21^o, how high is the hill?

Compound Angles

The formulae that follow are the ones we refer to as the compound angles

Sin (A + B) = sin A Cos B + Cos A sin B

Sin (A – B) = sin A Cos B – Cos A sin B

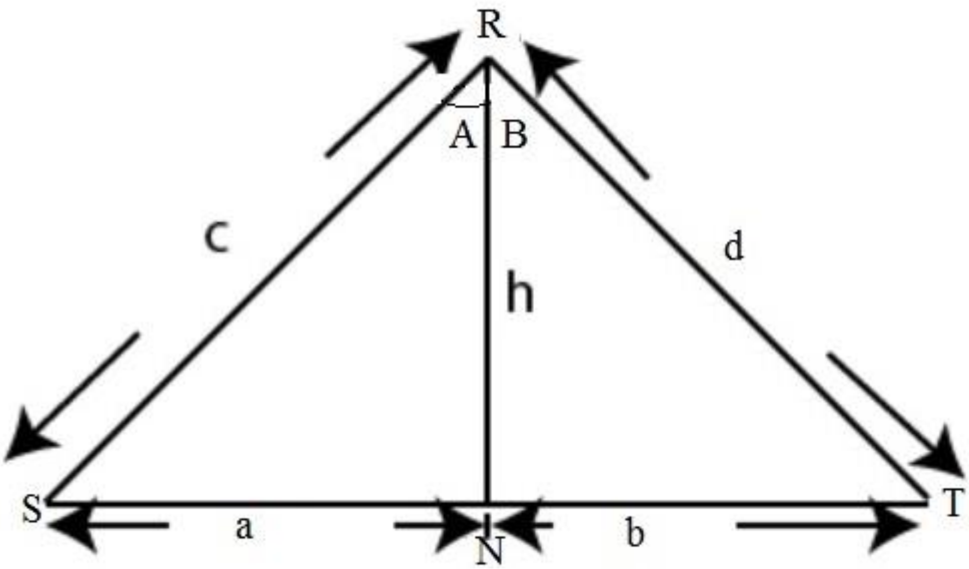
Cos (A + B) = Cos A Cos B – sin A sin B

Cos (A – B) = Cos A Cos B + sin A sin B

Solution

Sin (A + B) = Sin A Cos B + Cos A Sin B

Holder the area of the triangle RST



Area (ΔRST) = area (ΔRNS) + area (ΔRNT)

$\frac{1}{2}$cd sin (A + B) = $\frac{1}{2}$ hc sin B + $\frac{1}{2}$ hd sin B

Multiplied throughout $\frac{2}{cd}$

Sin (A + B) = $\frac{h}{d}$ sin A + $\frac{h}{d}$ sin B

But $\frac{h}{d}$ = cos B and $\frac{h}{c}$ = cos A

Sin (A + B) = sin A Cos B + Cos A sin B

Cos (A + B)

Apply the cosine rule on the same triangle in figure 8.16

$(a + b)^2 = c^2 + d^2 - 2cd \text{ Cos } (A + B)$

$$\text{Cos } (A + B) = \frac{c^2 + d^2 - (a+b)^2}{2cd}$$

$$= \frac{c^2 + d^2 - (a^2 + 2ab\ b^2)}{2cd}$$

$$= \frac{c^2 + d^2 - a^2 - 2ab - b^2)}{2cd}$$

$$= \frac{(c^2 - a^2) + (d^2 - b^2) - 2ab}{2cd}$$

$$= \frac{h^2 + h^2 - 2ab}{2cd}$$

$$= \frac{2h^2 - 2ab}{2cd}$$

$$= \frac{2h^2}{2cd} - \frac{2ab}{2cd}$$

$$= \frac{h}{c} \cdot \frac{h}{d} - \frac{a}{c} \cdot \frac{b}{d}$$

$= \frac{h}{c}$ = Cos A and $\frac{h}{d}$ = cos B

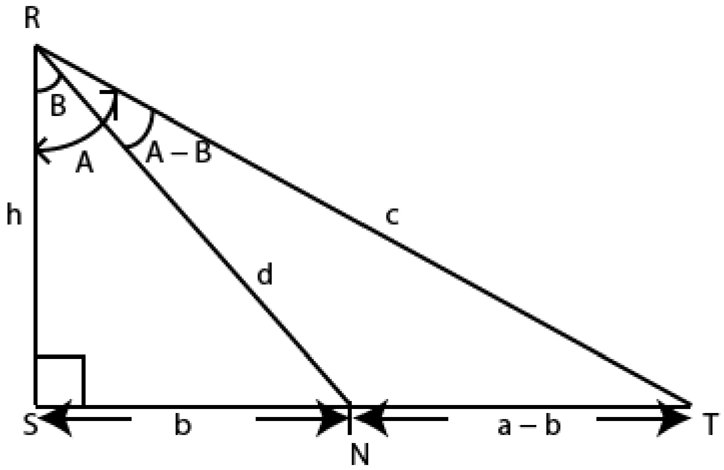
$= \frac{a}{c}$ = sin A and $\frac{b}{d}$ = sin B

Thus,

Cos (A + B) = Cos A Cos B – sin A sin B

Sin (A – B)

Consider the area of triangle RST



Area (ΔRNT) = area (ΔRST) – area (ΔRSN)

$\frac{1}{2}$ cd sin (A –B) = $\frac{1}{2}$ hc sin A - $\frac{1}{2}$ hd sin B

Multiplied throughout by $\frac{2}{cd}$

Sin (A – B) = $\frac{h}{d}$, sin A - $\frac{h}{c}$sin B

But $\frac{h}{d}$ = cos B, $\frac{h}{c}$ = cos A

Sin (A – B) = Cos B sin A – Cos A sin B

Cos (A – B) use the same figure and apply the cosine rule $(a – b)^2 = c^2 + d^2 – 2cd\text{Cos}(A – B)$

$$\text{Cos}(A + B) = \frac{c^2 + d^2 - (a - b)^2}{2cd}$$

$$= \frac{c^2 + d^2 - a^2 + 2ab - b^2}{2cd}$$

$$= \frac{(c^2 + d^2) + (d^2 + b^2) + 2ab}{2cd}$$

$= \frac{h^2 + h^2 + 2ab}{2cd}$

$= \frac{2h^2 + 2ab}{2cd}$

$= \frac{h^2}{cd} + \frac{ab}{cd}$

$= \frac{h}{c} \cdot \frac{h}{d} + \frac{a}{c} \frac{b}{d}$

But, $\frac{h}{c}$ = Cos A, $\frac{h}{d}$ = Cos B, $\frac{b}{d}$ = Sin B

Cos (A – B) = Cos A Cos B + Sin A Sin B

Tan (A + B) = $\frac{\sin(A+B)}{\cos(A+B)}$

$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

Divide numerator and denominator by Cos A Cos B

$$\frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{C\cos A \cos B}}$$

$$\therefore \text{Tan } (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Tan (A – B) = $\frac{\sin(A-B)}{\cos(A-B)}$

$$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

Divide numerator and denominator by Cos A Cos B

=

$$\frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \cos A \sin B}$$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\therefore \text{Tan } (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Example 1.9

1. Without using tables, evaluate the following

a) Tan (195^o)

b) Sin 15^o

c) Cos 75^o

d) Tan 15^o

Solution

a) Tan 195^o = tan ($135^o + 60^o$) = $\frac{\tan 135^o - \tan 60^o}{1 + \tan 135^o \tan 60^o}$

$$= \frac{-1 - \sqrt{3}}{1 - \sqrt{3}}$$

b) Sin 15^o = sin (45^o 30^o) = sin 45^o Cos 30^o – Cos 45^o sin 30^o

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$

$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

c) Cos 70° = Cos (45° + 30°) = Cos 45° Cos 30° – sin 45° sin 30°

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$

$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

d) Tan 45° = tan (45° – 30°)

$$= \frac{\tan 45^o - \tan 30^0}{1 + \tan 45^o \tan 30^o}$$

$$= 1 - \frac{\sqrt{3}}{3} = \frac{\frac{3 - \sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$$

$$= \frac{3-\sqrt{3}}{3+\sqrt{3}}$$

Example 1.10

1. Prove that Cosec θ = Sin θ + Cos θ cot θ

Solution

Take the RHS

Using cot θ = $\frac{\cos\theta}{\sin\theta}$

Cosec θ = sin θ + Cos θ x $\frac{\cos\theta}{\sin\theta}$

= $\frac{\sin\theta}{1} + \frac{\cos^2\theta}{\sin\theta}$

= $\frac{\sin^2\theta + \cos^2\theta}{\sin\theta}$ (common denominator)

= $\frac{1}{\sin\theta}$ ($\sin^2\theta + \cos^2\theta$)

= Cosec θ

RHS = LHS, hence proved

Solution

Take the RHS

$$\text{Cot}\,\Theta - 1 = \frac{1 - \tan\Theta}{\tan\theta}$$

$$= \frac{\tan\Theta}{1 - \tan\Theta}$$

= $\frac{\tan\theta}{1+\tan\theta}$

= $\frac{1-\tan\theta}{1+\tan\theta}$

RHS = LHS, hence proved

Exercise 1.11

1. Simplify a) $\sin^4\theta - \cos^4\theta$

   b) $\frac{1-\sin 2\theta}{\cos\theta}$

   c) $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$

   d) $\frac{1+\cos 2\theta}{\cot^2\theta}$

2. Provide the identities

   a) $\tan\theta \cot\theta \sec\theta \cos\theta = 1$

   b) $\frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta}$

   c) $\frac{1}{\sin\theta} = \frac{1}{1+\sin\theta} = 2\tan\theta\sec\theta$

   d) $\tan\theta + \cot\theta = \sec\theta\cos\theta = 1$

   e) $(a\cos\theta + b\sin\theta)^2 + (-a\sin\theta + b\cos\theta)^2 = a^2 + b^2$

   f) $(\operatorname{cosec}\theta - \cot\theta)^2 = \frac{1-\cos\theta}{1+\tan\theta}$

   g) $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$

   h) $(\cos\theta - \sin\theta)^2 + (\operatorname{cosec}\theta + \sin\theta)^2 = 2$

3. Compute without tables or calculators

a) $\sin 15^o$ b) $\sin 75^o$ c) $\tan 15^o$ d) $\tan 75^o$ c) $\sin 195^o$

4. Prove the following identities

a) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

b) Sin (A +B) + sin (A – B) = 2sin A sin B

c) Cos (A +B) + cos (A – B) = 2cos A cos B

d) Cos (A +B) + cos (A – B) = 2sin A sin B

5. Simplify the following

a) Sin (A + 2πr)

b) Cos (A + π)

c) Sin (A + $\frac{\pi}{2}$)

d) Cos (A + 2π)

6. Express 6 sin (x + 60^o) in the form of p sin x + Q sin x

7. If cos A =$\frac{5}{13}$, tan B = $\frac{3}{5}$, A and B being acute. Evaluate the following

a) Cos (A + B)

b) Tan (A + B)

c) Sin (A + B)

Double angle formulae

By applying the knowledge of compound angles, it is clear that

(1). sin 2x = sin (x + x)

= sinx cosx + cosx sinx

Thus **sin 2x = 2sin x cos x**

(2). Cos 2x = cos x cos x – sin x sin x

$= \cos^2 x - \sin^2 x$

From $\cos^2 x + \sin^2 x = 1$

$\text{Sin}^2 x = 1 - \cos^2 x$

$\text{Cos}^2 x = \cos^2 x - (1 + \cos^2 x)$

$= \cos^2 x + \cos^2 x - 1$

$\text{Cos}2x = 2\cos^2 x - 1$

But $\cos^2 x = 1 - \sin^2 x$

$\text{Cos}^2 x - \sin^2 x = \cos 2x$

$1 - \text{Sin}2\, x - \sin^2 x = \cos 2x$

$1 - 2\sin^2 x = \cos^2 x$

$\therefore \text{Cos}^2 x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

3. Tan 2x = tan (x + x)

$= \frac{\tan x + \tan x}{1 - tan^2 x}$

$\text{Tan } 2x = \frac{2 \tan x}{1 - tan^2 x}$

<u>Exercise 1.12</u>

1. a) Express sin 3x in terms of cos x

   b) Express cos 3x in terms of cos x

   c) Express tan 3x in terms of tan x

2. Use double angle formulae to prove that

   a) $\text{Sin } 4x = 8\cos^3 x \sin x - 4\cos x \sin x$

   b) $\text{Cos } 4x = 8\cos^4 x - 8\cos^2 x + 1$

How to write cos 2A and sin 2A in terms of tan A

From double angle formula, it is clear that,

a) Cos 2A = $\frac{\cos^2 A - \sin^2 A}{1}$

From $\cos^2 A + \sin^2 A = 1$

Cos 2A = $\frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$

Divide the numerator and denominator by cos ^{2} A we have

$$\frac{\frac{\cos^2 A}{\cos 2^A} - \frac{\sin^2 A}{\cos^2 A}}{\frac{\cos^2 A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

b) Similarly, from double angle formula of sine, we see that

Sin 2A = $\frac{2 \sin A \cos A}{1}$

= $\frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A}$

Dividing numerator and denominator by $\cos^2$ A, we have

$$\text{Sin } 2A = \frac{\frac{2\sin A \cos A}{\cos^2 A}}{\frac{\text{Cos}^2 A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}}$$

Sin 2A = $\frac{2\tan A}{1+\tan^2 A}$

Example 1.13

Solve sin (2^{π} + θ) + cos (θ - $\frac{\pi}{2}$) = 1, where $0^o \leq \theta \leq 360^o$

Solution

Cos 2^{π} = 1, sin 2^{π}= 0, cos $\frac{\pi}{2}$ = 0 and sin $\frac{\pi}{2}$ = 1

Expand the left part of the equation i.e sin (2^{π} + θ) + cos (θ - $\frac{\pi}{2}$)

Sin 2^{π}cos θ + cos 2^{π}sin θ + cos θ cos $\frac{\pi}{2}$ + sin θ sin $\frac{\pi}{2}$ = 1

Sin θ + sin θ = 1

2sin θ = 1

Sin θ = $\frac{1}{2}$

θ = 30^o, 150^o

Exercise 1.14

1. Solve the following trigonometric equations for $0 \leq \theta \leq 360^o$

    a) Sin θ - 1 = 0

    b) 2cos θ = $\sqrt{2}$

    c) Tan θ = $\frac{\sqrt{3}}{3}$

    d) Tan 3θ = 1

    e) 2cos θ

    f) Cos θ = sin θ

g) Sin θ = 2

2. Solve for θ, where $0^o \leq \theta \leq 360^o$

a) $\cos^2 \theta - 2\cos \theta = 0$

b) $2\sin^2 \theta + \sin \theta - 1 = 0$

c) 2cos 2 + 3cos θ + 1 = 0

d) 4sin2 + 4sin θ = 3

3. Solve for θ, where $0^o \leq \theta \leq 360^o$

a) cos (sin θ) = 1

b) cos 2θ + sin θ - 1

c) 2sin 2θ - tan θ = 0

d) Sin 4θ cos 2θ cos 4θ sin 2θ = $\frac{\sqrt{2}}{2}$

CIRCLE

Length of an Arc

Consider the following

$$\frac{\text{length of arc AB}}{2\pi r} = \frac{\Theta}{2\pi}$$

2π (length of arc AB) = θ x $2\pi r$

Length of arc AB = rθ

Area of a circular sector

Pg. 141 drawing

$$\frac{Area\ of\ sector\ A\ O\ B}{Area\ of\ circle} = \frac{\theta}{360^0}$$

Where: area of circle = πr^2 and 360° = 2π

$$\frac{area\ of\ sector\ AOB}{\pi r^2} = \frac{\theta}{2\pi}$$

Area of sector AOB x 2π = θ x πr^2

$$\text{Area of sector AOB} = \frac{\Theta\ x\ \pi r^2}{2\pi}$$

Area of sector AOB = $\frac{1}{2}r^2\theta$

Where θ is in radians and r is the radius of the circle

Example 1.15

Find the arc length of a circle of radius r which is subtended by a central angle θ for each of the following

a) r = 3cm and θ = 45°

b) r = 6cm and θ = 180°

Use π = 3.14

Solution

a) 1 = rθ

1 = 3 x $\frac{\pi}{3}$cm or $\frac{3\ x\ 3.14}{4} = \frac{9.42}{4}$ = 2.354 cm

b) arc length, 1 = r θ

$1 = 6 \times \pi$

$1 = 6 \times 3.14$

$= \underline{18.84\text{cm}}$

Example 1.16

Find the area of the sector of a circle with radius 6cm and subtended by an angle of 60^o use $\pi = 3.14$

Solution

Area $= \frac{1}{2} r^2\theta$

$= ½ \times 6 \times 6 \frac{\pi}{r}$

$= 6\pi\ \text{cm}^2$

Since $\pi = 3.14$

Therefore, area $= 3.14 \times 6$

$= 18.84\text{cm}^2$

Trigonometric Equation

Example 1.17

Solve $\sin \theta = ½$ for $0^o \leq \theta \leq 360^o$

Solution

$\text{Sin } \theta = ½$

$\theta = \text{Sin}^{-1}(½)$

$\theta = 30^o, 150^o$

Example 1.18

Solve 2sin θ - $\sqrt{3}$ = 0 for 0 ≤ θ ≤ 360

Solution

2 sin θ = $\sqrt{3}$

Sin θ = $\frac{\sqrt{3}}{2}$

θ = 60°, 120°

Example 1.19

$2\cos^2\theta + 3\cos\theta - 2 = 0$

Solution

This equation is quadratic in cos θ

Factorize the equation

$2\cos^2\theta + 3\cos\theta - 2 = 0$

Let cos θ = t

$2t^2 + 3t - 2 = 0$

$2t^2 + 4t - t - 2 = 0$

$2t(t + 2) - (t + 2) = 0$

$(t + 2)(2t - 1) = 0$

t = -2 or 2t = 1

t = -2 or t = ½

But, t = Cos θ

Cos θ = -2

θ = undefined or no solution since the minimum value of cos θ = -1

Cos θ = ½ = θ = 60°, 300°

θ = (60°, 300°)

Example

Solve Cos 2 θ Cos θ + Sin 2θ = 1, where 0≤ θ ≤ 360°

Solution

Use the compound angle formula Cos (A – B) = Cos A Cos B + Sin A Sin B

Thus, Cos 2 θ Cos θ + Sin θ = Cos (2θ - θ) = Cos θ

Cos θ = 1

θ = (0°, 360°)

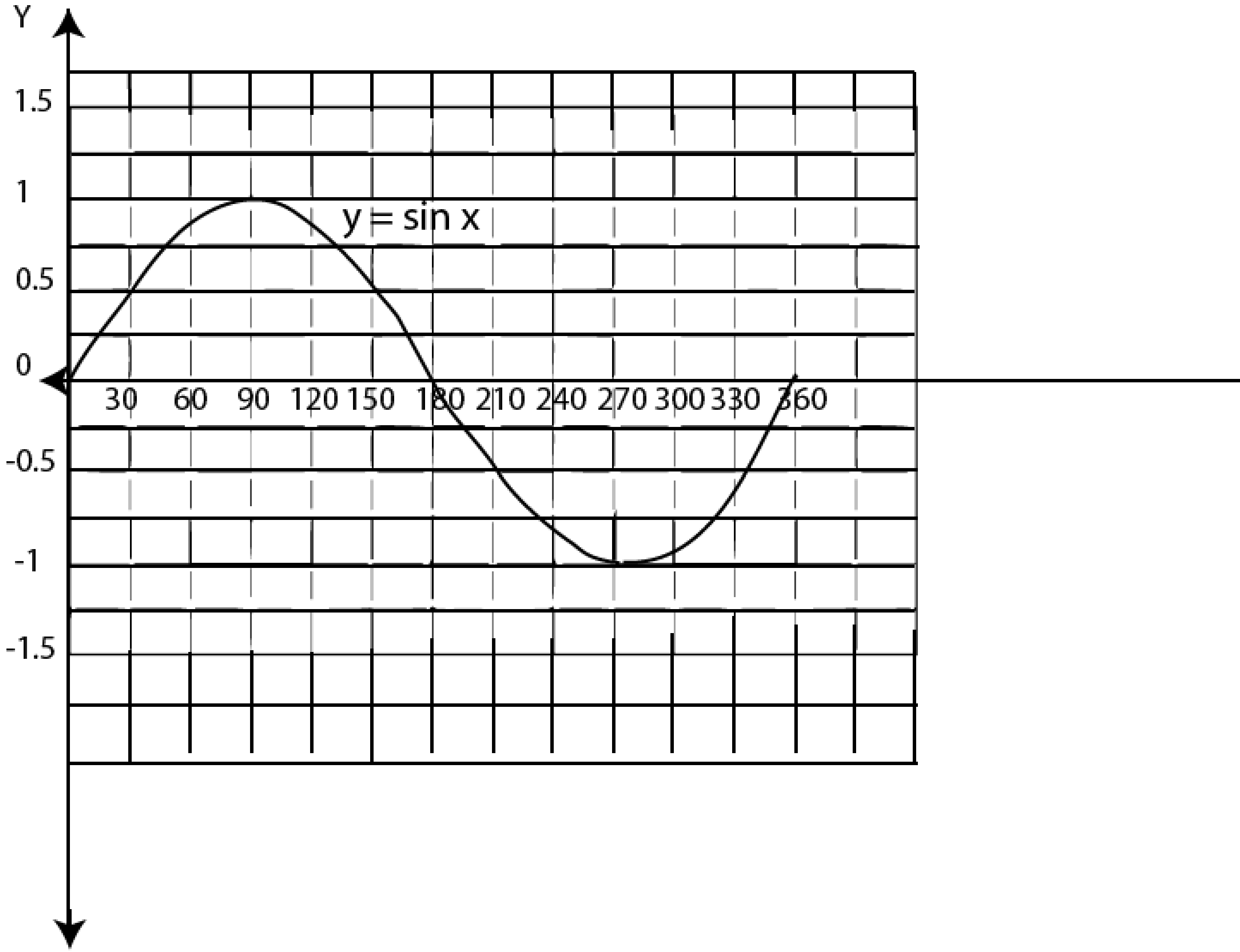
Trigonometric functions

Let x be angle in degree or radians and f (x) be the image of x. Then the functions defined as f (x) = sin x, f (x) = cos x and f (x) =tan are called trigonometric functions. These functions can be graphed just like any other functions. Values of angles should be on the horizontal axis and images values should be on the vertical axis.

Sine functions f (x) = sin x

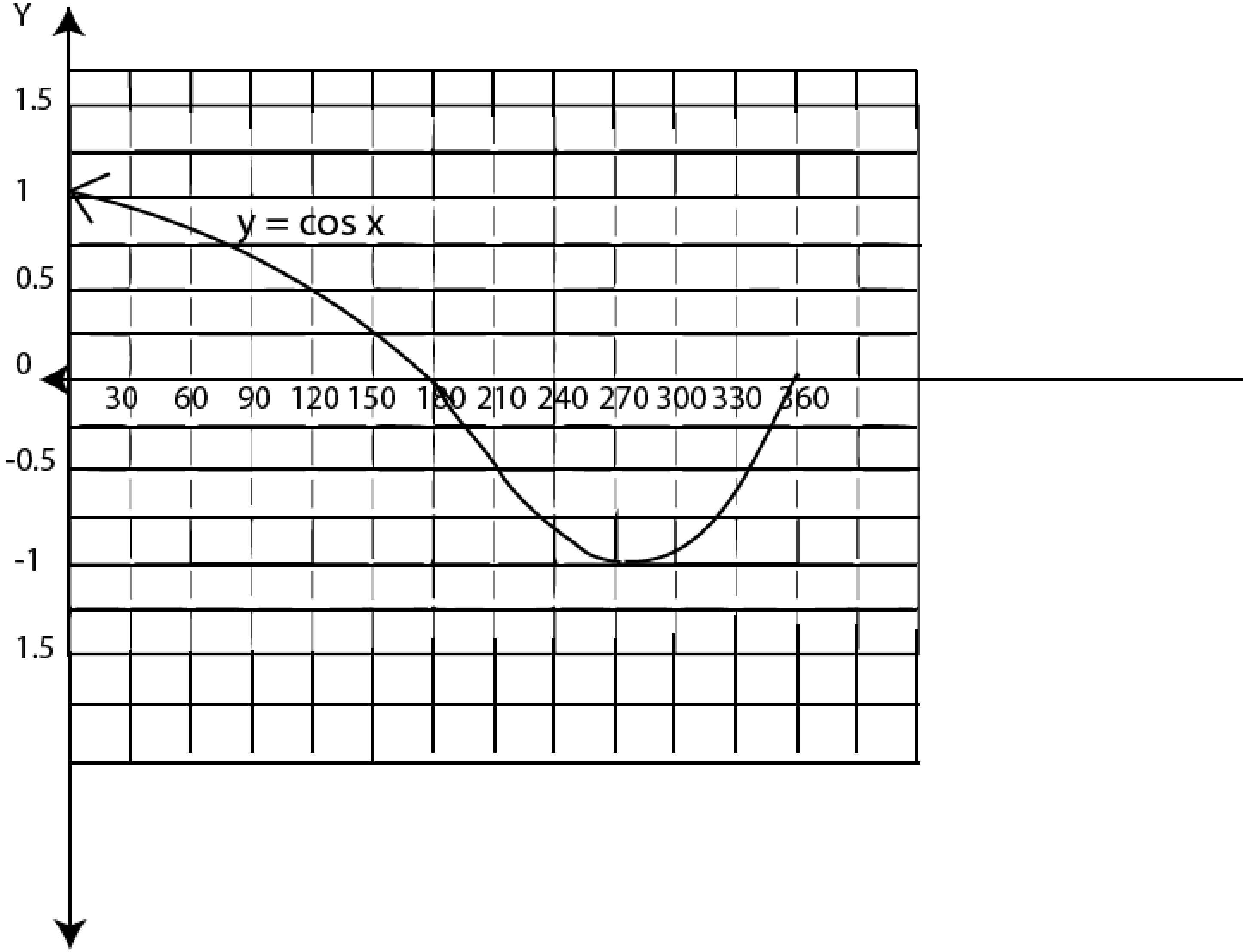
To graph sine functions, draw a table of values in which x values will be angles and y = f (x) will be images of these angles. The angles may be in degrees or radians. Table 8.1 has values that suit a function y = sin x, (The y – values are correct to one decimal place)

x	0	30	45	60	90	120	135	150	180	210	225	240	270
ysinx	0	0.5	0.7	1	1	0.9	0.7	0.5	0	-0.5	-0.7	-0.9	-1



Cosine function y = cos x

To graph cosine function follow the procedure followed in drawing sine functions. Table 8.2 contains values for y = cos x. In table 8.2 values are given to one decimal place.



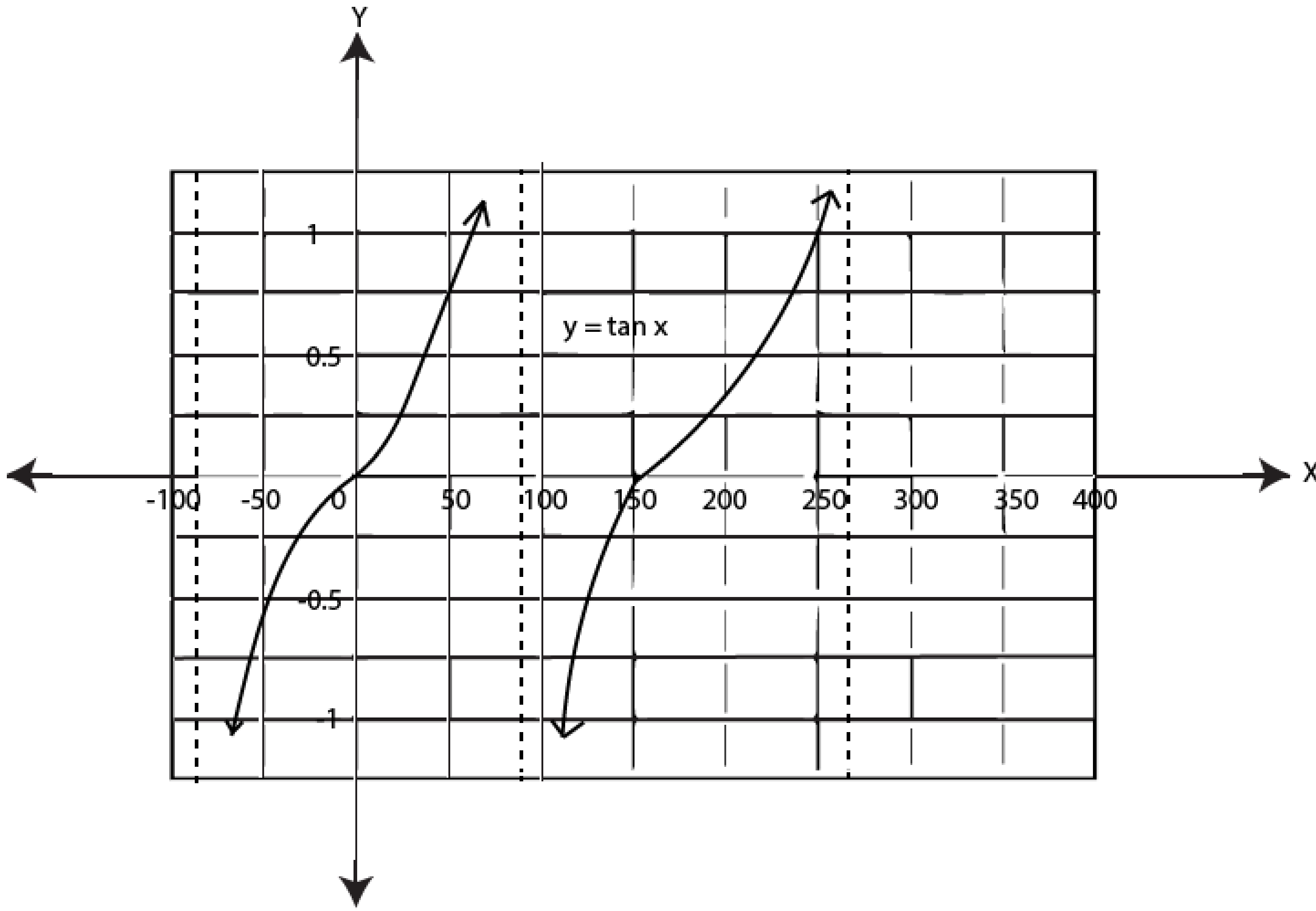
Note

1. The domain of y = sin x and y = cos x is set of real numbers and range is (-1 ≤ y ≤ 1).
2. These functions are periodic, they repeat after every one complete revolution (360^o)

Graph of y = tan x

To graph y = tan x draw a table values and plot the point required then join the points with given correct to one decimal places.

x^n	90	60	- 45	30	0	30	45	60	90	120	135	150	180
Y tan x		1.7	- 30	-1	0	-0.6	1	1.7	∝	- 1.7	-1	-0.6	0



Note

1. The range of y = tan x is a set of real numbers

2. The domain is (x: x ≠ 90^o, x ≠ 270, x ≠ 450…)

3. The tangent function is periodic too. It repeats itself after every 180^o

4. At x = 90^o, x = 450^o etc, the functions not defined. These values are its asymptotes. It tends to approach these values but never touches them

Exercise

1. State the range and period for each of the following

a) y = cos x

b) y = ½ cos 2x

c) y = ½ cos x

d) y = 3 sin x

e) y = 3 – sin x

j) y = 1 + sin $\frac{x}{2}$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

BASIC PROPERTIES OF EXPONENTIAL FUNCTIONS

If a is a real number and a is greater than 0 (a>0) then;

F(x) = a^x is called an exponential function

Examples:

1. f (x) = 2^x
2. f (x) = 3^x
3. f (x) = e^x
4. f(x)= 5^x

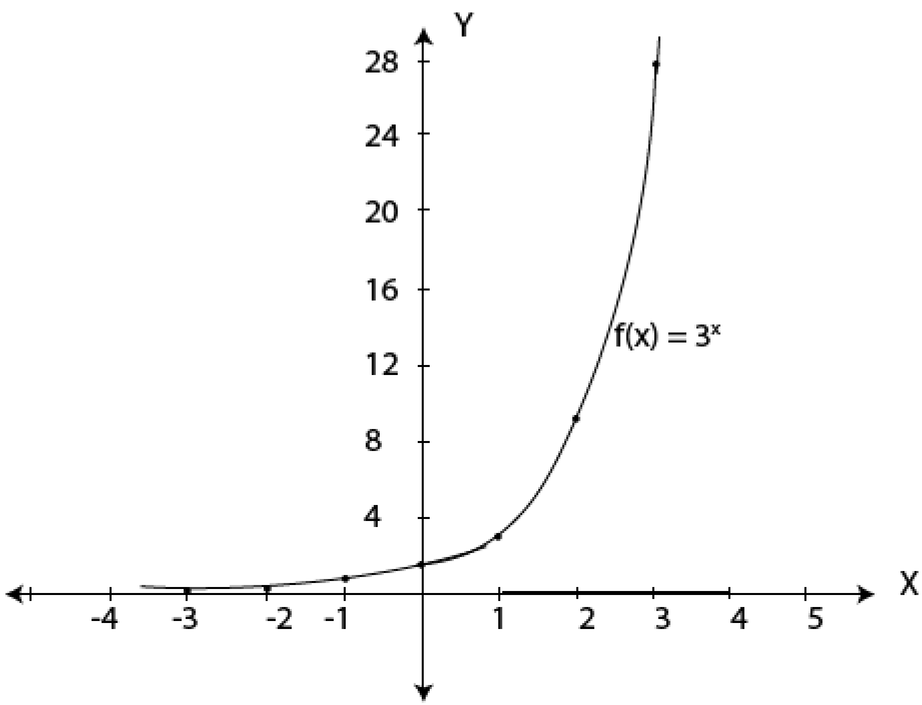
Graphs of Exponential Function:

Example:

Draw the graph of f(x) = 3^x

Solution:

X	-3	-2	-1	0	1	2	3
F(x)	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27



The properties of exponential functions

-Any Function of the form f (x) = a^x has the following properties

i. Increases when x is (+ve)

ii. Decrease when x is (-ve)

iii. It is constant when x=1

iv. It's graph passes through (0,1)

v. It is one-to-one function

vi. It's domain is {x: x∈IR}

vii. It's range is {y: y is positively}

viii. It's horizontal asymptotes at y=0

Roles of Exponential Functions

1. If n ΣN then a^n is the product of N

This means;

If a^5 = a x a x a x a x a

But a^0=1

2. If n,m ΣN, then

i. $a^{n/m} = \sqrt[m]{a^n} = (\sqrt[m]{a})^n$

ii. $a^{-n} = \frac{1}{a^n}$

iii. $a^n a^m = a^{(n+m)}$

iv. $\frac{a^n}{a^m} = a^{(n-m)}$

V. $(a^n)^m = a^{nm}$

Example

Solve $4^{x-3} = 8$

Express them in terms of base 2

$2^{2(x-3)} = 2^3$

2(x-3) = 3

2x-6=3

2x = 3 + 6

$\frac{2x}{2} = \frac{9}{2}$

X = 4.5

EXERCISE

1. Draw graphs of the following functions and determine the domain and range.

   a) $f(x) = 2^{x-2}$

   b) $f(x) = 2 + 2^x$

   c) $f(x) = 4^{-x}$

2. Solve for x

   a) $27^{(x+1)} = 9$

   b) $25^{(x-3)} = 125^x$

   c) $4^{(5x+1)} = 16^{2x+1}$

   d) $(1-x)^5 = (2x-1)^5$

   e) $9^x = 3^{3x-1}$

THE CALCULUS OF EXPONENTIAL FUNCTIONS

A) The derivative of $f(x) = e^x$

Recalling that:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots$

If y = e^x

$\frac{dy}{dx} = \frac{d}{dx}(e^x)$

= $\frac{d}{dx}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots$

= $0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots\ldots$

BASIC PROPERTIES OF LOGARITHMIC FUNCTIONS

If a is any positive real number which in not equal to 1, then a function f (x) defined as f (x) = log_a^x for x> 0 is called a logarithmic function.

Examples:

i. log_2^x

ii. log_3^x

iii. log_e^x

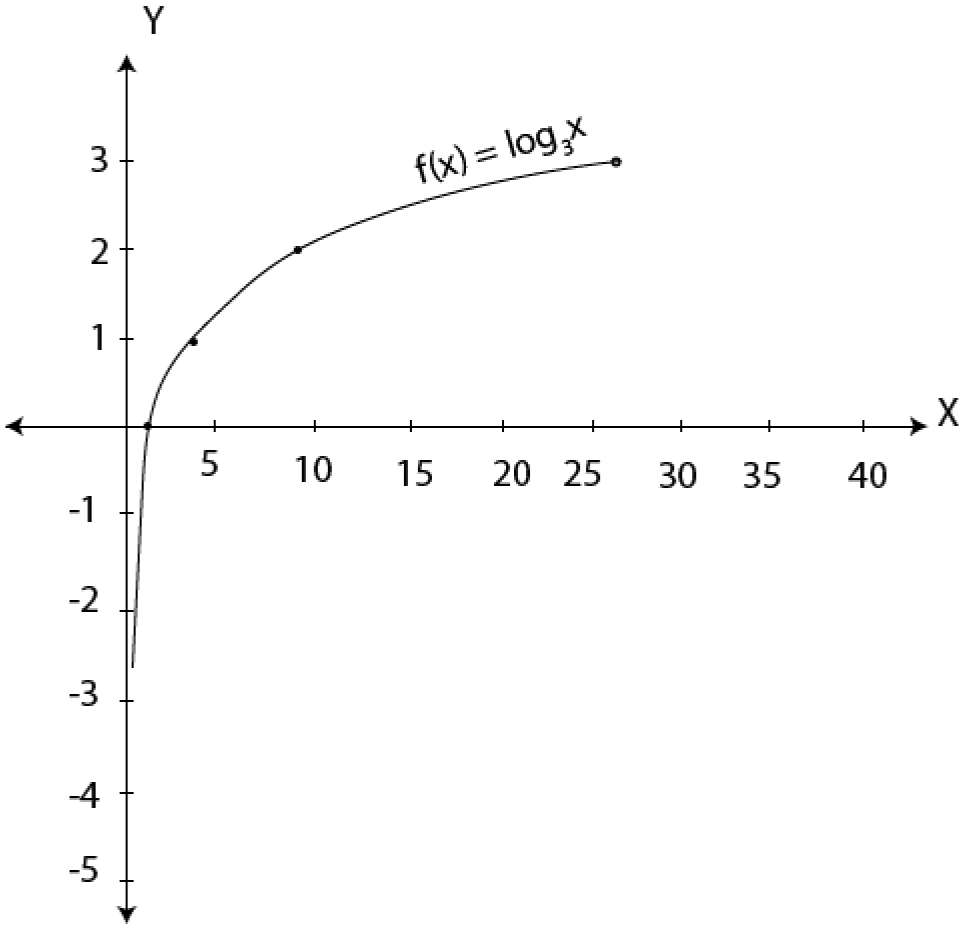
iv. log_{10}^x

Graphs of logarithmic functions

Draw graph of f (x) = log_3^x

Solution

x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81
f (x)	-3	-2	-1	0	1	2	3	4



Properties of logarithmic functions

i. It's graph passes through (1,0)

ii. It's domain is (x : x > 0)

iii.It's range is (y:y∑IR)

iv.It's a one to one function

v. It is the inverse of exponential function

vi.It's vertical asymptote is at x = 0

vii.No horizontal asymptote

LAWS OF LOGARITHMS:

$i) log_a^{xy}$ = log_a^x + log_a^y

ii). $log_a^{\frac{x}{y}}$ = log_a^x - log_a^y

$iii). log_a^{x^n}$ = n log_a^x

$iv). log_a^{a^n}$ = n

$v)\ log_a^1$ = 0

$vi)\ log_a^a$ = 1

$vii)\ a^{log_a^x}$ =x , x > 0

CHANGING BASES OF LOGARITHM

Let y = log_a^N

Change it into base b

Steps:

i. Express logarithmic function into exponential function

Y = log_a^N ………… (i)

N = a^y ……………. (ii)

Introduce logarithm to base b both sides:

N = a^y

log_b^N = $log_b^{a^y}$

log_b^N = y log_b^a ………….. (iii)

Divide (iii) by log_b^a ……….. (iii)

$\therefore$ y = $\frac{log_b^N}{log_b^a}$

Examples

Evaluate the following to four decimal places

1. $log_3^{5.2}$

2. $log_{0.5}^{0.0372}$

Solution

1. $log_3^{5.2}$

Let y = $log_3^{5.2}$

3^y = 5.2

Introducing natural logarithm

In 3^y = In 5.2

yln 3 = in 5.2

$\frac{yln3}{ln3}$ = $\frac{\ln 5.2}{\ln 3}$ = $\frac{1.6487}{1.0986}$

$\therefore$ y = 1.5007

2. $log_{0.5}^{0.0372}$

Let y = $log_{0.5}^{0.0372}$

0.5^y = 0.0372

ln 0.5^y = ln 0.0372

$\frac{y ln 0.5}{ln 0.5} = \frac{\ln 0.0372}{\ln 0.5} = \frac{-3.2914}{-0.6931}$

$\therefore$ y = 4.7488

EXERCISE

1. Evaluate the following correct to 4 decimal places

   a) Log 49236

   b) Ln 54.02

   c) log_7^{13}

   d) $log_5^{120.24}$

   e) log_9^{72}

   f) $log_7^{304.66}$

2. Change to base e

a) Log 5.3147

b) Log 0.053

3. Change to base 10

a) ln2.0103

b) ln 47.486

4. Draw graphs of

a) f (x) = log_4^x

b) f (x) = log_2^x

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

a) Integration of sin x

Recalling;

$\frac{d}{dx}(\cos kx) = -k\sin kx$

$\int \sin kx dx = \frac{-1}{k}\cos kx + c$

Example

Compute

$$\int \sin 5x dx$$

Solution

Let u = 5x

Du = 5dx

$\therefore dx = \frac{du}{5}$

$\therefore \int \sin 5x dx = \frac{1}{5}\int \sin u du$

$= \frac{-1}{5}\cos u + c$

$= \frac{-1}{5}\cos 5x + c$

b) Integration of $\cos x$

Recalling

$\frac{d}{dx}(\sin kx) = k\cos kx$

$\therefore \int \cos kx dx = \frac{1}{k} \sin kx + c$

c) Integration of $\sec^2 x$

Recalling

$\frac{d}{dx} (\tan kx) = k \sec 2kx$

$\therefore$ Ê$f\sec^2 kxdx$ = $\frac{1}{k} \tan kx + c$

APPLICATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

There are numerous disciplines in which the exponential and logarithmic function are applicable, two of which are compound interest and depreciation.

Compound interest

The compound amount A of a principle P and interest at the end of n years at the rate r compound annually, is given by $A_n = P(1 + r)^n$

Note:

$$r = \frac{RT}{100}$$

where,

T - is the compounding period

R- is the rate of interest growth

The compound interest can be derived in a number of ways, In general if P is the principle, R is the rate (%) of interest I, then the amount A after T period in years will be,

$$A = P + I, \text{ Where } I = \frac{PRT}{100}$$

For year 1; It will be,

$$A_1 = P + \frac{PRT}{100} \quad \text{OR}$$

$$A_1 = P\left(1 + \frac{RT}{100}\right) \quad \text{.........................(i)}$$

For year 2: It will be,

$$A_2 = A_1 + \frac{A_1RT}{100} \quad \text{OR}$$

$$A_2 = A_1\left(1 + \frac{RT}{100}\right) \quad \text{.....................(ii)}$$

Substituting eqn (i) into (ii)

$$A_2 = P\left(1 + \frac{RT}{100}\right)\left(1 + \frac{RT}{100}\right)$$

$$A_2 = P\left(1 + \frac{RT}{100}\right)^2$$

If the acceleration is made up to year n, the total amount A_n will generally be given as,

$$A_n = P\left(1 + \frac{RT}{100}\right)^n$$

Example;

Suppose TZs 2000/= is invested for 5 years at the rate of 2% compounded annually.

a) Find the compound amount, S

B) Find the compound interest, I

Solution

a) From,

$$A_n = P\left(1 + \frac{RT}{100}\right)^n$$

Where, P = 2000, r = 2%, n = 5

Then, S = $2000(1 + 0.02)^5$

= $2000(1.02)^5$ = 2208.16

∴ The compound Amount is TZs, 2208.16

b) Compound Interest,

I = S - P

I = 2208.16 - 2000

I = 208.16

∴ The compound interest is TZs, 208.16

DEPRECIATION

Depreciation is a reduction in the value of an asset with time passage, as a result of wear and tear, age or obsolescence.

Example

A phone costs TZs 40000/=. Then phone loses 10% of its value during the first year and 25% of its value during the second year. It means that in year 1, the depreciation will be,

10% of TZs 40000.

$$= \frac{10}{100} \times 40{,}000$$

$$= 4{,}000$$

So, the phone will be TZs, 40000 - 4000 = 36000 = TZs 36000/=

In year 2, the depreciation will be 25% of 36000/=

$$= \frac{25}{100} \times 36000$$

$$= 9000$$

The phone value will be 36000 - 9000 = 27,000.
Therefore after 2 years, the phone worth TZs 27,000/=

If an asset depreciates at R% per annum, then from the preset or initial value P its value in the year 1 will be,

$$p\left(1 - \frac{R}{100}\right)$$

Each succession year the value is multiplied by the same factor $\left(1 - \frac{R}{100}\right)$, and therefore after t years the future value V will be,

$$V = p\left(1 - \frac{R}{100}\right)^t$$

This is the depreciation formula, where V is the future, R the rate of depreciation , P the present or initial value and t is the number of years.

Question

A laptop purchased for TZs 600,000. depreciation at a rate of 30% per annum. After how many years will the machine have a value of TZs 400,000?

USE OF EXPONENTIAL FUNCTIONS FOR COMPOUND INTEREST PROBLEMS

The compound interest problem can be described by using the exponential function. Recall the formula for compound interest , the amount A after t years with a principle P and per annum interest rate r, compound n times per year is

$$A = p\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = p\left(1 + \frac{1}{\frac{n}{r}}\right)^{nt}$$

If we let

$$\frac{n}{r} = m$$

implying that n = Mr then by substitution we have,

$$A = P\left(\left(1 + \frac{1}{M}\right)^{m}\right)^{rt}$$

As $M \to \infty$, then interest will be compounded continuously and then $\left(1 + \frac{1}{M}\right)^{m} = e$

This implies that as the interest gets compounded continuously, the amount A after t years will be exponential function with a continuously compounding interest growing exponentially given by,

$$A = Pe^{rt}$$

Where, P = Principle amount
r = annual interest rate
t = number of years.

MATRICES

is an arrangement of number in rows and columns. Its usually denoted using capital letters

E.g A = $\begin{pmatrix} 1 & 4 \\ 6 & 7 \end{pmatrix}$ B = $\begin{pmatrix} 4 & 1 & 3 \\ 2 & 8 & 1 \\ 3 & 0 & 7 \end{pmatrix}$ C = $\begin{pmatrix} 4 & 1 & 2 \\ 4 & 0 & 6 \end{pmatrix}$

Order: A matrix is said to be of order m x n if it has m rows and n columns.

E.g. Order of A is 2 x 2

Order of B is 3 x 3

Order of C is 2 x 3

Types

- **Rows matrix** is a matrix with only one row.

  e.g. D = (2 1 6)

- **Column matrix** is a matrix with only one column.

  E.g. Ð„ = $\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$

- **Square matrix** is a matrix with equal number of rows and column

  E.g. A = $\begin{pmatrix} 1 & 4 \\ 6 & 7 \end{pmatrix}$ B = $\begin{pmatrix} 4 & 1 & 3 \\ 2 & 8 & 1 \\ 3 & 0 & 7 \end{pmatrix}$

- **Identity matrix** (T) is a square matrix with all elements in the leading diagonal equals to 1 and the rest are 0.

  E.g. I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

  I = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Property

AI = A

Null (zero) matrix (Ƶ) is a matrix with all elements equation to 0 e.g. Ƶ = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Properties

1) A + Ƶ = A

2) AƵ = Ƶ

Operations

i) Addition/ subtraction

This is only possible if the matrices have the same order

E.g. A = $\begin{pmatrix} a & b & c \\ d & e & \boldsymbol{f} \\ g & h & k \end{pmatrix}$ B = $\begin{pmatrix} l & m & n \\ p & q & r \\ s & t & v \end{pmatrix}$

$$A + B = \begin{pmatrix} a + \mathrm{l} & b + \mathrm{m} & c + \mathrm{n} \\ d + \mathrm{p} & e + \mathrm{q} & \boldsymbol{f} + \mathrm{r} \\ g + \mathrm{s} & h + \mathrm{t} & k + \mathrm{v} \end{pmatrix}$$

ii) Scalar multiplication

Given a scalar multiplication then

A = t $\begin{pmatrix} a & h & c \\ d & e & f \\ g & h & h \end{pmatrix}$

= $\begin{pmatrix} ta & tb & tc \\ td & te & tf \\ tg & th & tk \end{pmatrix}$

E.g.: Evaluate tP – sQ

Given t = 3 s = ½, P = $\begin{pmatrix} 1 & 0 & 3 \\ 5 & 6 & 8 \\ 2 & 4 & 7 \end{pmatrix}$

Q = $\begin{pmatrix} -2 & 4 & 8 \\ 3 & -5 & 6 \\ 7 & 4 & 0 \end{pmatrix}$

<u>Solution</u>

tP – SQ

$$=3\begin{pmatrix} 1 & 0 & 3 \\ 5 & 6 & 8 \\ 2 & 4 & 7 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -2 & 4 & 8 \\ 3 & -5 & 6 \\ 7 & 4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 9 \\ 15 & 18 & 24 \\ 6 & 12 & 21 \end{pmatrix} - \begin{pmatrix} -1 & 2 & 4 \\ 3/2 & -5/2 & 3 \\ 7 & 2 & 0 \end{pmatrix}$$

$$\therefore tP - sQ = \begin{pmatrix} 4 & -2 & 5 \\ 27/2 & 41/2 & 21 \\ 5/2 & 10 & 21 \end{pmatrix}$$

iii) <u>Product</u>

AB is only possible if the order of A is m x n and that of B is n x p and the resulting matrix will be of the order m x p.

e.g. i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}$

2 x 2 2 x 1

$\begin{pmatrix} ae & + & bf \\ ce & + & df \end{pmatrix}$

ii) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix}$

$$\begin{pmatrix} al & bm & cn \\ dl & em & fn \\ gl & hm & kn \end{pmatrix}$$

E.g. $\begin{pmatrix} 3 & 1 & 2 \\ 6 & 0 & 4 \\ 5 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

3 x 3 3 x 1

$$= \begin{pmatrix} 3\,x\,1 + 1\,x\,2 + 2\,x\,3 \\ 6\,x\,1 + 0\,x\,2 + 4\,x\,3 \\ 5\,x\,1 + -1\,x\,2 + 2\,x\,3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 + & 2 + & 6 \\ 6 + & 0 + & 12 \\ 5 + & -2 + & 6 \end{pmatrix} = \begin{pmatrix} 11 \\ 18 \\ 9 \end{pmatrix}$$

iv) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} l & m & n \\ p & q & r \\ s & t & r \end{pmatrix}$

3 x 3 3 x 3

$$= \begin{pmatrix} al + bp + cs & am + bq + ct & an + br + cr \\ dl + ep + fs & dm + eq + ft & dn + er + fr \\ gl + hp + ks & gm + hq + ht & gn + hr + hr \end{pmatrix}$$

E.g. $\begin{pmatrix} 3 & 1 & 2 \\ 6 & 0 & 4 \\ 5 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 & 8 \\ 2 & -3 & 4 \\ 3 & 2 & 1 \end{pmatrix}$

3 x 3 3 x 3

$$\begin{pmatrix} 3\,x1+1\,x\,2+2\,x\,3 & 3\,x\,4+1\,x\,3+2\,x\,2 & 3\,x\,8+1\,x\,4+2\,x\,1 \\ 6\,x\,1+0\,x\,2+4\,x\,3 & 6\,x\,4+0x-3+4\,x\,2 & 6\,x\,8+0+4+4\,x\,1 \\ 5\,x\,1+\ -1\,x\,2+2\,x\,3 & 5\,x\,4+\ -1\,x-3+2\,x\,2 & 5\,x\,8+\ -1\,x\,4+2\,x\,1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 13 & 30 \\ 18 & 32 & 52 \\ 9 & 27 & 38 \end{pmatrix}$$

Determinant of a 2 x 2 matrices

Given A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Determinant of A = $|A|$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

= ad – cb

If the determinant A = 0 then A is singular matrix

E.g. which of the following is singular or non singular matrix

A = $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

$$|A| = (2 \text{ x } 4) - (3 \text{ x } 3)$$

8 – 9 = -1 Non - singular matrix

B = $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$

$|B|$ = 1(6) – 2(3)

= 6 – 6

= 0 Singular matrix

C = $\begin{vmatrix} -3 & -1 \\ -1 & -3 \end{vmatrix}$

$|C| = -3\,(-3) - -1\,(-1)$

$|C| = 9 - 1$

= 8 Non – singular matrix

<u>Inverse of a 2 x 2 matrix</u>

Given A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The inverse of A = A^{-1}

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

E.g. A = $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

$$\mathbf{A^{-1}} = \frac{1}{|A|}\text{Aadj}$$

$|A|$ = 2 (4) – 3(3) = -1

$$\text{Adj} = \begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-1}\begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{4}{-1} & \frac{-3}{1} \\ \frac{-3}{-1} & \frac{2}{-1} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$$

Solving system of simultaneous equations in 2 unknowns

Given the following equations

$$a_1x + b_1y = p$$
$$a_2x + b_2y = q$$

In a matrix form

$$\begin{pmatrix} a_1 & b_1 \\ d_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Determinant method

1. **(Cramer's rule)**

Solving for x

$$x = \frac{\Delta_x}{\Delta}$$ where

$\Delta_x = \begin{vmatrix} p & b_1 \\ q & b_2 \end{vmatrix}$ and $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Solving for y

$y = \frac{\Delta y}{\Delta}$ where

$\Delta_y = \begin{vmatrix} a_1 & p \\ a_2 & q \end{vmatrix}$ and $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

E.g. solving the following using Cramer's rule

x + y = 3

x – y = 1

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

A= $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ Δ = $|A|$ = 1 (-1) – (1)= -2

$x = \frac{\Delta_x}{\Delta}$ where $\Delta x = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = 3(-1) - 1(1) = -4$

$x = \frac{-4}{-2} = 2$

$y = \frac{\Delta y}{\Delta}$

Δ = $|A|$ = 1 (-1) – (1)= -2

$\Delta y = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1(1) - 1(3) = -2$

$y = \frac{-2}{-2} = 1$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2. **Inverse method**

$\begin{pmatrix} a_1 & b_1 \\ d_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} p \\ q \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{Aadj}$$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} b2 & -b1 \\ -d2 & a1 \end{pmatrix}$$

E.g. x + y = 3

x – y = 1

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

A = $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $|A|$ = -1(1) – 1 (1) = -2

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$A^{-1} = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{-2} & \frac{1}{-2} \end{pmatrix}$

$A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} {}^{1}/_{2}\ x\ 3 + & {}^{1}/_{2}\ x 1 \\ {}^{1}/_{2}\ x\ 3 + & -{}^{1}/_{2}\ x\ 1 \end{pmatrix}$

$= \begin{pmatrix} \frac{3}{2} + & \frac{1}{2} \\ \frac{3}{2} + & -\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} \frac{4}{2} \\ \frac{2}{2} \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Determinant of a 3 x 3 matrix

Given $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$= a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$|A|$ = a (ek – fh) – b (dk – fg) + c (dh – eg)

E.g Find $|B|$ given B = $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \\ 5 & -2 & 3 \end{pmatrix}$

$$|B| = \begin{vmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \\ 5 & -2 & 3 \end{vmatrix}$$

$= 3 \begin{vmatrix} 1 & 4 \\ -2 & 3 \end{vmatrix} - (-1) \begin{vmatrix} 0 & 4 \\ 5 & 3 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ 5 & -2 \end{vmatrix}$

= 3 (3 + 8) + 1(0 – 20) + 2 (0 – 5)

= 33 – 20 - 10

= 3

$|B|$ = 3

Transpose

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ b_{12} & b_{22} & b_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}$$

$$A^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

E.g, $B = \begin{pmatrix} 3 & 0 & 5 \\ -1 & 1 & -2 \\ 5 & -2 & 3 \end{pmatrix}$

$B^T = \begin{pmatrix} 3 & \mathbf{-1} & 5 \\ \mathbf{0} & 1 & -2 \\ \mathbf{5} & \mathbf{-2} & 3 \end{pmatrix}$

Co factors

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$

A co factor of an element in a 3 x 3 matrix is given by determine determinant of a 2 x 2 matrix which is formed by removing elements in the same row and column with given element and multiply by 1 or -1 according to the following procedure.

Produce $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

E.g. C = $\begin{pmatrix} 3 & 0 & 5 \\ -1 & 1 & -2 \\ 2 & 4 & 3 \end{pmatrix}$

$\text{Cof } 3 = +1 \begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

= (3 + 8) = 11

Cof 0 $-1 \begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix}$

= -1 (-3 + 4) = -1

$\text{Cof } 5 = +1 \begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}$

= + (-4 – 2) = -6

Cof (-1) $-1 \begin{vmatrix} 0 & 5 \\ 4 & 3 \end{vmatrix}$

= -1 (0 – 20) = 20

Cof 1 + 1 $\begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix}$

= 1 (9 – 10) = -1

COf -2

$-1 \begin{vmatrix} 3 & 0 \\ 2 & 4 \end{vmatrix}$

= -1 (12 – 0) = -12

Cof 2

$1 \begin{vmatrix} 0 & 5 \\ 1 & -2 \end{vmatrix}$

= 1 (0 – 5) = -5

Cof 4

$-1\begin{vmatrix} 3 & 5 \\ -1 & -2 \end{vmatrix}$

= -1 (-6 + 5) = 1

Cof 3

$1\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix}$

= 1 (3 + 0) = 3

Matrix of Cofactors

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

$$\text{Matrix of cofactors} = \begin{pmatrix} Cof\ a & Cof\ b & Cof\ c \\ Cof\ d & Cof\ e & Cof\ f \\ Cof\ g & Cof\ h & Cof\ k \end{pmatrix}$$

Adjoin of A = adj, A^T = the transpose of matrix of Cofactors

$$\text{Adj } A = \begin{pmatrix} cof\ a & cof\ d & cof\ g \\ cof\ b & cof\ e & cof\ h \\ cof\ c & cof\ f & cof\ k \end{pmatrix}$$

Inverse of A

$$A^{-1} = \frac{1}{|A|} A_{adj}$$

Summary

Procedure for finding inverse of a 3 x 3 matrices

i) Find the determinant

ii) Find the cofactors

iii) Form the matrix of cofactors

iv) Form the adjoint

v) Find the value

Solving simultaneous equation in 3 unknown

1. Determinant method (Cramer's rule)

Given that $a_1x + by + c_{1z} = p$

$a_2x + b_2y\ c_2z = q$

$a_3x + b_3y\ c_3z = r$

In a matrix form we have

$$\begin{pmatrix} a_1 & b_1 & z_1 \\ a_2 & b_2 & z_2 \\ a_3 & b_3 & z_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$x = \frac{|B|}{|A|}, B = \begin{pmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{pmatrix}$$

$$y = \frac{|C|}{|A|}, C = \begin{pmatrix} a_1 & p & c_1 \\ a_2 & q & c_2 \\ a_3 & r & c_3 \end{pmatrix}$$

$$z = \frac{|D|}{|A|}, D = \begin{pmatrix} a_1 & b_1 & p \\ a_2 & b_2 & q \\ a_3 & b_3 & r \end{pmatrix}$$

E.g. using Cramer's rule Solve x + 2y + z = 6
2x + y – z = 3
3x – y + 2z = 5

Solution

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}$$

$$|A| = 1\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

= 1 (2 – 1) – 2 (4 + 3) + 1 (-2 – 3)

= 1 – 14 + -5 = -18

$$|B| = \begin{vmatrix} 6 & 2 & 1 \\ 3 & 1 & -1 \\ 5 & -1 & 2 \end{vmatrix}$$

$$|B| = 6\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 5 & -1 \end{vmatrix}$$

= 6 (2 – 1) – 2 (6 + 5) + 1 (-3

= 6 – 22 + - 8

$|B|$= -24

$$C = \begin{pmatrix} 1 & 6 & 1 \\ 2 & 3 & -1 \\ 3 & 5 & 2 \end{pmatrix}$$

$|C| = 1 \begin{vmatrix} 3 & -1 \\ 5 & 2 \end{vmatrix} - 6 \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}$

= 1 (6 + 5) – 6 (4 + 3) + 1 (10 – 9)

= 11 – 42 + 1

$|C|$ = -31 + 1 = -30

$$D = \begin{pmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \\ 3 & -1 & 5 \end{pmatrix}$$

$|D| = 1 \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + 6 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$

= 1 (5 + 3) – 2 (0 – 9) + 6 (-2 – 3)

= 8 – 2 + - 30

= 6 – 30

$|D|$ = -24

$$x = \frac{|B|}{|A|} = \frac{-24}{-18} = \frac{4}{3}$$

$$y = \frac{|C|}{|A|} = \frac{-30}{-18} = \frac{5}{3}$$

$$z = \frac{|D|}{|A|} = \frac{-24}{-18} = \frac{4}{3}$$

II. Inverse method

$a_1x + b_1y + c_1z = p$

$a_2x + b_2y + c_2z = q$

$a_3x + b_3y + c_3z = r$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$AA^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$I \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

E.g: solving using inverse method

$x + 2y + z -= 6$

$2x + y – z = 3$

$3x – y + 2z = 5$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix}$$

$|A| = 1\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$

= 1 (2 – 1) – 2 (4 + 3) + 1 (-2 – 3)

= 1 – 14 + (-5)

= -13 – 5

$|A|$ = -18

A = $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix}$

Cof 1 = + 1 $\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$ cof 2 = -1 $\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$

= +1 (2 – 1) = 1 (4 + 3)

<u>= 1</u> <u>= -7</u>

Cof 1 = + 1 $\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}$ cof 2 = -1 $\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$

= + (-2 – 3) = -1 (4 + 1)

<u>= -5</u> <u>= -5</u>

Cof 1: + 1 $\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$ cof -1 = -1 $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$

= +1 (2 – 3) = -1 (-1 – 6)

<u>= -1</u> <u>= 7</u>

Cof 3: + 1 $\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$ cof -1= -1 $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$

= + 1 (-2 – 1) = -1 (-1 – 2)

<u>= -3</u> <u>= 3</u>

Cof 2: + 1 $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$

= + 1 (1 – 4)

= -3

Matrix of cofactors, C

$$C = \begin{pmatrix} 1 & -7 & -5 \\ -5 & -1 & 7 \\ -3 & 3 & -3 \end{pmatrix}$$

Note: Adj A = C^T

$$\text{Adj } A = \begin{pmatrix} 1 & -5 & -3 \\ -7 & -1 & 3 \\ -5 & 7 & -3 \end{pmatrix}$$

Inverse of A

$$A^{-1} = \frac{1}{|A|} A_{adj}$$

$$A^{-1} = \frac{1}{-18} \begin{pmatrix} 1 & -5 & -3 \\ -7 & -1 & 3 \\ -5 & 7 & -3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1/18 & 5/18 & 3/18 \\ 7/18 & 1/18 & -3/18 \\ 5/18 & -7/18 & 3/18 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/18 & 5/18 & 3/18 \\ 7/18 & 1/18 & -3/18 \\ 5/18 & -7/18 & 3/18 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{-18}x6 + & \frac{5}{18}x\,3 + & \frac{3}{18}x5 \\ \frac{7}{18}x\,6 + & \frac{1}{18}x\,3 + & -\frac{3}{18}x\,5 \\ \frac{5}{18}x\,6 + & -\frac{7}{18}x\,3 + & \frac{3}{18}x\,5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{18} + & \frac{15}{18} + & \frac{15}{18} \\ \frac{42}{18} + & \frac{3}{18} + & -\frac{15}{18} \\ \frac{30}{18} + & -\frac{21}{18} + & \frac{15}{18} \end{pmatrix}$$

$$= \begin{pmatrix} -6 + 15 + 15 \\ 18 \\ 42 + 3 - 15 \\ 18 \\ 30 - 21 + 15 \\ 18 \end{pmatrix} = \begin{pmatrix} \frac{24}{18} \\ \frac{30}{18} \\ \frac{24}{18} \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \\ \frac{4}{3} \end{pmatrix}$$

Exercise

1. a) Given A = $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, and B = $\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$, Find A + B

   b) If A = $\begin{pmatrix} 2 & k \\ k & 8 \end{pmatrix}$ is singular find the value of k

c) Find the inverse of B = $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$

d) Solve $\begin{cases} x + y = 10 \; using \; inverse \\ 2x - y = 5 \quad\quad method \end{cases}$

2. Using i) Determinant ii) Inverse, solve for x, y and z

3x – y + z = 2

x + 5y + 2z = 6

2x + 3y + z——————= 0

3. a) solve $\begin{cases} 2x + y = 8 \\ 4x - y = 10 \end{cases}$

i) Using determinant
ii) inverse

b) If A = $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$, B = $\begin{pmatrix} 4 & -5 \\ 9 & 7 \end{pmatrix}$ and

C = $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$

Show that (A + B – 2C) is singular

c) Solve x + y + z-= 6

3x – 2y – z-= -1

2x + 4y + 3z = 19 Using i) determinant

ii) Inverse

4. Solve, 2x – 3y + z = 3

-x + 4y + 3z = 16

3x + 2y – 2z = 1

5. A transformation is given by the matrix M where M = $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ Find the (a) image of (-2, 5) under M (b) Inverse of M.

6. If T is linear transformation such that T = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and T (x, y) = (3y, 5x), Find T hence evaluate T (1, 2)

b) Use inverse method to solve

$$\begin{cases} 2y + 3x - 15 = 0 \\ 2x - 20 + 3x = 0 \end{cases}$$

7. a) Given A = $\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$. B = $\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$, Find AB and BA

b) Find the value of x,y,w and z

$$3 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2w \end{pmatrix} + \begin{pmatrix} 4x + & y \\ z + w & 3 \end{pmatrix}$$

SOLUTION

1. a) A = $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ B = $\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$

A + B

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$$

$= \begin{pmatrix} 2 + -1 \\ -1 + 5 \\ 3 + -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$

$\therefore$ A + B = $\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$

$A = \begin{pmatrix} 2 & k \\ k & 8 \end{pmatrix}$

For singular matrix, $|A|$ = 0

$|A| = \begin{vmatrix} 2 & k \\ k & 8 \end{vmatrix}$

= 2 (8) – k (k) = 0

16 = k^2 = 0

16 = k^2

$\sqrt{16} = \sqrt{k^2}$

4 = k

$\therefore$ <u>k = 4</u>

c) B = $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$

$$B^{-1} = \frac{1}{|B|}\begin{pmatrix} -1 & -1 \\ -4 & 2 \end{pmatrix}$$

$|B|$ = 2 (-1) – 4 (1)

= -2 – 4

$|B|$ = -6

$$B^{-1} = \frac{1}{-6}\begin{pmatrix} -1 & -1 \\ -4 & 2 \end{pmatrix}$$

$$B^{-1} = \frac{1}{-6}\begin{pmatrix} 1/6 & 1/6 \\ 2/3 & -1/3 \end{pmatrix}$$

d) By using inverse method required to solve for x and y
x + 2y = 10

2x – y = 5

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

A = $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

$|A|$ = 1 (-1) – 2 (2) = -1 – 4 - -5

A^{-1} = 1/ $|A|$ $\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}$

A^{-1} = 1/-5 $\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}$

$$A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}\begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{5}\ x\ 10 + \frac{2}{5}\ x\ 5 \\ \frac{2}{5} x\ 10 + -\frac{1}{5} x\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{10}{5} + \frac{10}{5} \\ \frac{20}{5} + -\frac{5}{5} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{20}{5} \\ \frac{15}{5} \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

2. 3x – y + 2z = 2

   x – 5y + 2z = 6

   2x + 3y + z = 0

   i) By determinant

$$\begin{pmatrix} 3 & -1 & 2 \\ 1 & 5 & 2 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 5 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

$$|A| = 3\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} - {}^{-}1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}$$

= 3 (5 – 6) - -1 (1 – 4) + 2 (3 – 10)

= 3 ($^{-}$1) + 1 (-3) + 2 ($^{-}$7)

= -3 + -3 + -4 = -20

B = $\begin{pmatrix} 2 & -1 & 2 \\ 6 & 5 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

$|B|$ = 2 $\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix}$ - 1 $\begin{vmatrix} 6 & 2 \\ 0 & 1 \end{vmatrix}$ + 2 $\begin{vmatrix} 6 & 5 \\ 0 & 3 \end{vmatrix}$

= 2 (5 – 6) + 1 (6 – 0) + 2 (18 – 0)

= 2 ($^-$1) + 1 (6) + 2 (18)

= -2 + 6 + 36 = 40

C = $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 6 & 2 \\ 2 & 0 & 1 \end{pmatrix}$

$|C|$ = 3 $\begin{vmatrix} 6 & 2 \\ 0 & 1 \end{vmatrix}$ - 2 $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ + 2 $\begin{vmatrix} 1 & 6 \\ 2 & 0 \end{vmatrix}$

= 3 (6 – 0) – 2 (1 – 4) + 2 (0 – 12)

= 3 (6) – 2 ($^-$3) + 2 ($^-$12)

= 18 – 6 + -24 = -12

D = $\begin{pmatrix} 3 & -1 & 2 \\ 1 & 5 & 6 \\ 2 & 3 & 0 \end{pmatrix}$

$|D| = 3\begin{vmatrix} 5 & 6 \\ 3 & 0 \end{vmatrix}$ - $^-$1 $\begin{vmatrix} 1 & 6 \\ 2 & 0 \end{vmatrix} + 2 \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}$

= 3 (0 – 18) + 1 (0 – 12) + 2 (3 – 10)

= 3 ($^-$18) + -1 ($^-$12) + 2 ($^-$7)

$|D| = -54 - 12 - 14 = {}^{-}80$

$X = \frac{|B|}{|A|}$

$= \frac{40}{-20} = \frac{20}{-10} = -2$

$Y = \frac{|C|}{|A|}$

$= \frac{-12}{-20} = \frac{16}{10} = \frac{3}{5}$

$Z = \frac{|D|}{|A|}$

$= \frac{-80}{-20} = \frac{40}{10} = 4$

ii) By inverse

$$\begin{pmatrix} 3 & -1 & 2 \\ 1 & 5 & 2 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$$

$|A| = 3\begin{vmatrix} 5 & -2 \\ 3 & 1 \end{vmatrix} - 1\ {}^{-}1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}$

= 3 (5 – 6) + 1 (1 – 4) + 2 (3 – 10)

= 3 (-1) + 1 (-3) + 2 (-7)

= -3 + -3 + -14 = -20

A = $\begin{pmatrix} 3 & -1 & 2 \\ 1 & 5 & 2 \\ 2 & 3 & 1 \end{pmatrix}$

Cof 3: + 1 $\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix}$ cof -1: 1 $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$

= + 1 (5 – 6)   = - 1 (1 – 4)

= -1   = 3

Cof 2: + 1 $\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}$ cof: - 1 $\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix}$

= + 1 (3 – 10)   = -1 (5 – 6)

= -7   = 1

Cof 2: + 1 $\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$ cof2: -1 $\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$

= + 1 (3 – 4)   = -1 (9 – 2)

= -1   = -7

Cof 4 + 1 $\begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix}$ cof 3: -1 $\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$

= + 1 (-2 – 10)   = -1 (6 – 2)

= -12   = -4

Cof 1: + 1 $\begin{vmatrix} 3 & -1 \\ 1 & 5 \end{vmatrix}$

= + 1 (15 + 1)

= 16

Matrix of the Co factors

$$C = \begin{pmatrix} -1 & 3 & -7 \\ 1 & -1 & -7 \\ -12 & -4 & 16 \end{pmatrix}$$

$$\text{Adj } A = \begin{pmatrix} -1 & 1 & -12 \\ 3 & -1 & -4 \\ -7 & -7 & 16 \end{pmatrix}$$

Inverse A

$$A^{-1} = \frac{1}{|A|} A_{adj}$$

$$= \frac{1}{-20} \begin{pmatrix} -1 & 1 & -12 \\ 3 & -1 & -4 \\ -7 & -7 & 16 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{20} & -\frac{1}{20} & \frac{12}{20} \\ -\frac{3}{20} & \frac{1}{20} & \frac{4}{20} \\ \frac{7}{20} & \frac{7}{20} & \frac{16}{20} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{20} & -\frac{1}{20} & \frac{12}{20} \\ -\frac{3}{20} & \frac{1}{20} & \frac{4}{20} \\ \frac{7}{20} & \frac{7}{20} & \frac{16}{20} \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{20} x\ 2 + -\frac{1}{20} x\ 6 + \frac{12}{20} x\ 0 \\ -\frac{3}{20} x\ 2 + \frac{1}{20} x\ 6 + \frac{4}{20} x\ 0 \\ \frac{7}{20}\ x\ 2 + \frac{7}{20} x\ 6 + -\frac{16}{20} x\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{20} + -\frac{6}{20} \\ \frac{6}{20} + \frac{6}{20} \\ \frac{14}{20} + \frac{42}{20} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{4}{-20} \\ \frac{0}{20} \\ \frac{56}{20} \end{pmatrix}$$

3. a) $2x + y = 8$

   $4x - y = 10$

i) Determinant

$A = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$

$|A|$ = 2 (-1) – 4 (1)

= -2 – 4 = -6

$|B| = \begin{vmatrix} 8 & 1 \\ 10 & -1 \end{vmatrix}$

$|B|$ = 8 (-1) – 10 (1)

= -8 – 10 = -18

$|C| = \begin{vmatrix} 2 & 8 \\ 4 & 10 \end{vmatrix}$

$|C|$ = 2 (10) – 4(8)

= 20 – 32 = -12

$X = \frac{|B|}{|A|}$

$= \frac{-18}{6} = 3$

$Y = \frac{|C|}{|A|} = \frac{-12}{-6} = 2$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

X = $\frac{|B|}{|A|}$

ii) By inverse

A = $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$

$|A|$ = 2 (-1) – 4 (1)

= -12 – 4 = -6

$A^{-1} = \frac{1}{|A|} A_{adj}$ But

$A_{adj} = \begin{pmatrix} -1 & -1 \\ -4 & 2 \end{pmatrix}$

$A^{-1} = \frac{1}{6}\begin{pmatrix} -1 & -1 \\ -4 & 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/6 & 1/6 \\ 4/6 & -2/6 \end{pmatrix}\begin{pmatrix} 8 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{6}x\,8 + \frac{1}{6}x\,10 \\ \frac{4}{6}x\,8 + -\frac{2}{6}x\,10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{8}{6} + \frac{10}{6} \\ \frac{32}{6} + -\frac{20}{6} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{18}{6} \\ \frac{12}{6} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

b) If A= $\begin{pmatrix} 5 & 7 \\ -2 & 3 \end{pmatrix}$, B = $\begin{pmatrix} 4 & -5 \\ 9 & 7 \end{pmatrix}$ and C = $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$, Show that A + B – 2C is singular

$$\begin{pmatrix} 5 & 7 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & -5 \\ 9 & 7 \end{pmatrix} - 2\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$$

$= \begin{pmatrix} 5+4 & 7+-5 \\ -2+9 & 3+7 \end{pmatrix} - \begin{pmatrix} 4 & -6 \\ 2 & 2 \end{pmatrix}$

$= \begin{pmatrix} 9 & 2 \\ 7 & 10 \end{pmatrix} - \begin{pmatrix} 4 & -6 \\ 2 & 2 \end{pmatrix}$

$= \begin{pmatrix} 9-4 & 2—6 \\ 7-2 & 10-2 \end{pmatrix}$

$= \begin{pmatrix} 5 & 8 \\ 5 & 8 \end{pmatrix}$

$|A + B - 2C| = 5(8) - 5(8) = 0$ Hence shown.

c) x + y + z = 6

3x – 2y – z = -1

2x + 4y + 3z = 19

i) Determinant

$$\text{Let, } A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & -2 & -1 \\ 2 & 4 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & -2 & -1 \\ 2 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 19 \end{pmatrix}$$

$|A| = 1\begin{vmatrix} -2 & -1 \\ 4 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 2 & 4 \end{vmatrix}$

= 1 (-6 + 4) – 1 (9 + 2) + 1 (12 + 4)

= -2 – 11 + 16 = 3

B = $\begin{pmatrix} 6 & 1 & 1 \\ -1 & -2 & -1 \\ 19 & 4 & 3 \end{pmatrix}$

$|B|$ = 6 $\begin{vmatrix} -2 & -1 \\ 4 & 3 \end{vmatrix}$ - 1 $\begin{vmatrix} -1 & -1 \\ 19 & 3 \end{vmatrix}$ + 1 $\begin{vmatrix} -1 & -2 \\ 19 & 4 \end{vmatrix}$

= 6 (-6 + 4) – 1 (-3 + 19) + 1 (-4 + 38)

= 6 (-2) – 1 (16) + 1 (34)

= -12 – 16 + 34 = 6

C = $\begin{pmatrix} 1 & 6 & 1 \\ 3 & -1 & -1 \\ 2 & 19 & 3 \end{pmatrix}$

$|C|$ = 1 $\begin{vmatrix} -1 & -1 \\ 19 & 3 \end{vmatrix}$ - 6 $\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$ + 1 $\begin{vmatrix} 3 & -1 \\ 2 & 19 \end{vmatrix}$

= 1 (-3 + 19) – 6 (9 + 2) + 1 (57 + 2)

= 16 – 6 (11) + 1 (59)

= 16 – 66 + 59 = 9

D = $\begin{pmatrix} 1 & 1 & 6 \\ 3 & -2 & -1 \\ 2 & 4 & 19 \end{pmatrix}$

$|D|$ = 1 $\begin{vmatrix} -2 & -1 \\ 4 & 19 \end{vmatrix}$ - 1 $\begin{vmatrix} 3 & -1 \\ 2 & 19 \end{vmatrix}$ + 6 $\begin{vmatrix} 3 & -2 \\ 2 & 4 \end{vmatrix}$

= 1 (-38 + 9) – 1 (57 + 2) + 6 (12 + 9)

= 1 (-34) – 1 (59) + 6 (16)

= -34 – 59 + 96 = 3

$$X = \frac{|B|}{|A|} = \frac{6}{3} = 2$$

$$Y = \frac{|C|}{|A|} = \frac{9}{3} = 3$$

$$Z = \frac{|D|}{|A|} = \frac{3}{3} = 1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

ii) By inverse

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & -2 & -1 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 19 \end{pmatrix}$$

$$|A| = 1 \begin{vmatrix} -2 & -1 \\ 4 & 3 \end{vmatrix} - 1 \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + 1 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

= 1 (-6 + 4) – 1 (9 + 2) + 1 (12 + 4)

= 1 (-2) – 1 (11) + 1 (16)

= -2 – 11 + 16 = 3

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & -2 & -1 \\ 2 & 4 & 3 \end{pmatrix}$$

Cof 1: + 1 $\begin{vmatrix} -2 & -1 \\ 4 & 3 \end{vmatrix}$

= + 1 (-6 + 4)

<u>= -2</u>

Cof1: + 1 $\begin{vmatrix} 3 & -2 \\ 2 & 4 \end{vmatrix}$

= + 1 (12 + 4)

<u>= 16</u>

Cof2: + 1 $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

= + 1 (3 – 2)

<u>= 1</u>

Cof2: + 1 $\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}$

= + 1 (-1 + 2)

<u>= + 1</u>

Cof 3: + 1 $\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}$

= + 1 (-2 – 3)

<u>= -5</u>

cof1: + 1 $\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$

= -1 (9 + 2)

<u>= -11</u>

cof 3: -1 $\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix}$

= -1 (3 – 4)

<u>= -1</u>

cof1: - $\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$

= - 1 (4 – 2)

<u>= -2</u>

cof 4: - 1 $\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}$

= -1 (-1 – 3)

<u>= 4</u>

Matrix cofactors

$$A = \begin{pmatrix} -2 & -11 & 16 \\ 1 & 1 & -2 \\ 1 & 4 & -5 \end{pmatrix}$$

Adj A = $\begin{pmatrix} -2 & 1 & 1 \\ -11 & 1 & 4 \\ 16 & -2 & -5 \end{pmatrix}$

$A^{-1} = 1/|A|$ Adj A

$A^{-1} = {}^{1}/3 \begin{pmatrix} -2 & 1 & 1 \\ -11 & 1 & 4 \\ 16 & -2 & -5 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2/3 & 1/3 & 1/3 \\ -1/3 & 1/3 & 4/3 \\ 16/3 & -2/3 & -5/3 \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} x\, 6 + -\frac{1}{3}\, x -1 + \frac{1}{3} x\, 19 \\ -\frac{11}{3} x\, 6 + \frac{1}{3} x -1 + \frac{4}{3}\, x\, 19 \\ \frac{16}{3} x\, 6 + -\frac{2}{3} x -1 + -\frac{5}{3} x\, 19 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{12}{3} + -\frac{1}{3} + \frac{19}{3} \\ -\frac{66}{3} + -\frac{1}{3} + \frac{76}{3} \\ \frac{96}{3} + \frac{2}{3} + -\frac{95}{3} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{6}{3} \\ \frac{9}{3} \\ \frac{3}{3} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

7. a) A = $\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ B = $\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$

Calculate i) AB

$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ x $\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$

= $\begin{pmatrix} 3\,x\,4 + 2\,x - 5 & 3\,x - 2 + 2\,x\,3 \\ 5\,x\,4 + 4\,x - 5 & 5\,x - 2 + 4\,x\,3 \end{pmatrix}$

= $\begin{pmatrix} 12 + -10 & -6 + 6 \\ 20 + -20 & -10 + 12 \end{pmatrix}$

= $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

ii) BA

A = $\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ B = $\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$

$\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$

= $\begin{pmatrix} 4\,x\,3 + \;-2\,x\,5 & 4\,x\,2 + \;-2\,x\,4 \\ -5\,x\,3 + 3\,x\,5 & -5\,x\,2 + 3\,x\,4 \end{pmatrix}$

= $\begin{pmatrix} 12 + \;-10 & 8 + \;-8 \\ -15 + 15 & -10 + 12 \end{pmatrix}$

= $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

b) **Find the value of x, y, w and z**

3 $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ = $\begin{pmatrix} x & 6 \\ 1 & 2w \end{pmatrix}$ + $\begin{pmatrix} 4 & x+y \\ z+w & 3 \end{pmatrix}$

$\begin{pmatrix} 3x & 3y \\ 3z & 3w \end{pmatrix}$ = $\begin{pmatrix} x+4 & 6+x+y \\ -1+z+w & zw+3 \end{pmatrix}$

3x – x = 4

<u>2x</u> = <u>4</u>
2 2
x = 2

3y – y = 6 +
2
2y = 6 + 2
<u>2y</u> =<u>8</u>
y = 4
3w = 2w + 3
3w – 2w = 3
w = 3
3z = -1 + z + w
3z – z = -1 + 3
<u>2z</u> =
<u>2</u>
2 2

z = 1

∴ x = 2, y = 4, z = 1, w = 3

5. A transformation is given by the matrix M where M = $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

Find the (a) image of (-2, 5) under M (b) Inverse of M

a) Let, $\begin{pmatrix} x' \\ y' \end{pmatrix}$ be the image under M

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4\,x - 2 + 1\,x\,5 \\ 2\,x - 2 + 3\,x\,5 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -8 + 5 \\ -4 + 15 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 \\ 11 \end{pmatrix}$$

b) M = $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

$|M|$ = 4 (3) – 2

= 12 – 2 = 10

$M^{-1} = {}^1/10 \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$

$$M^{-1} = \begin{pmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{2}{10} & \frac{4}{10} \end{pmatrix}$$

6. a) If T is linear transformation such that T = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and T (x, y)

(3y, 5x)

Find T hence evaluate T (1, 2)

Solution

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ 5x \end{pmatrix}$

$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} 3y \\ 5x \end{pmatrix}$

ax + by = 3y

a =0

b = 3

cx + dy = 5x

c = 5

d = 0

T = $\begin{pmatrix} 0 & 3 \\ 5 & 0 \end{pmatrix}$

T (x, y) = (3y, 5x)

T (1, 2) = (6, 5)

OR

$\begin{pmatrix} 0 & 3 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 0+6 \\ 5+0 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

LINEAR PROGRAMMING

This topic uses the knowledge of graphical solution of system of linear equations and inequalities. Owing to that, we will start this topic by looking at this important background knowledge.

SOLUTION OF SYSTEM OF LINEAR EQUATIONS

A system of linear equations can be solved graphically, by graphing the two linear equations on the same Cartesian coordinate plane, and solution of this is given by the point of intersection.

Example 11.1

Solve the system of equations by graphing $\begin{cases} 3x - y = 4 \\ 2x + y = 6 \end{cases}$

SOLUTION

We must obtain a table of values for the two linear equations. In all these, we will use only two points. This is because two points determine a straight line.

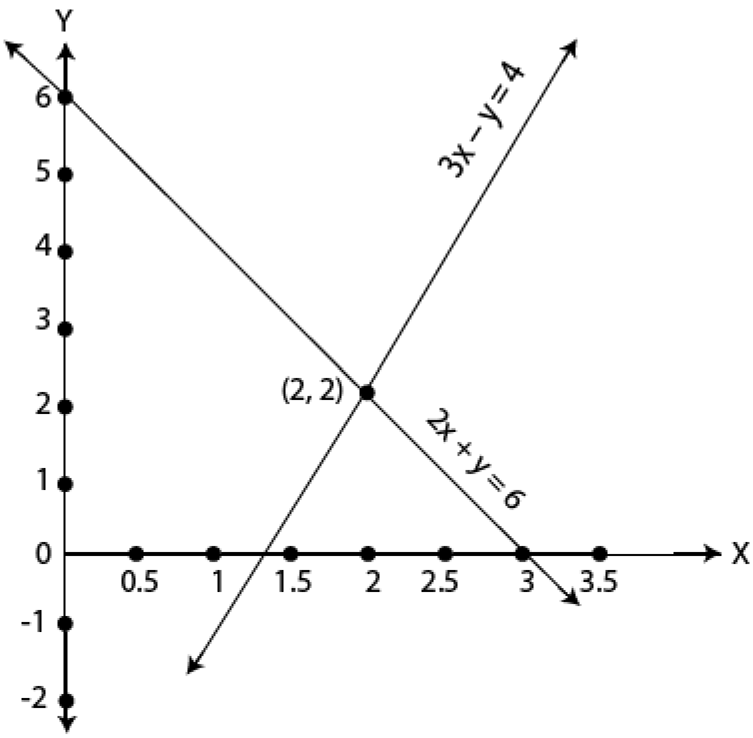
Table 11.1 3x – y = 4

X	0	4/3
Y	-4	0

Table 11.2 2x + y = 6

X	0	3
Y	6	0

Figure11.1 shows the graph of the information's in table 11.1 and 11.2



The two lines intersect at (2, 2)

Thus, the solution of the system is (2, 2) i.e x = 2 and y = 2

A system of linear equations that has unique solution like $\begin{cases} 3x - y = 4 \\ 2x + y = 6 \end{cases}$ is said to be

Consistent and independent

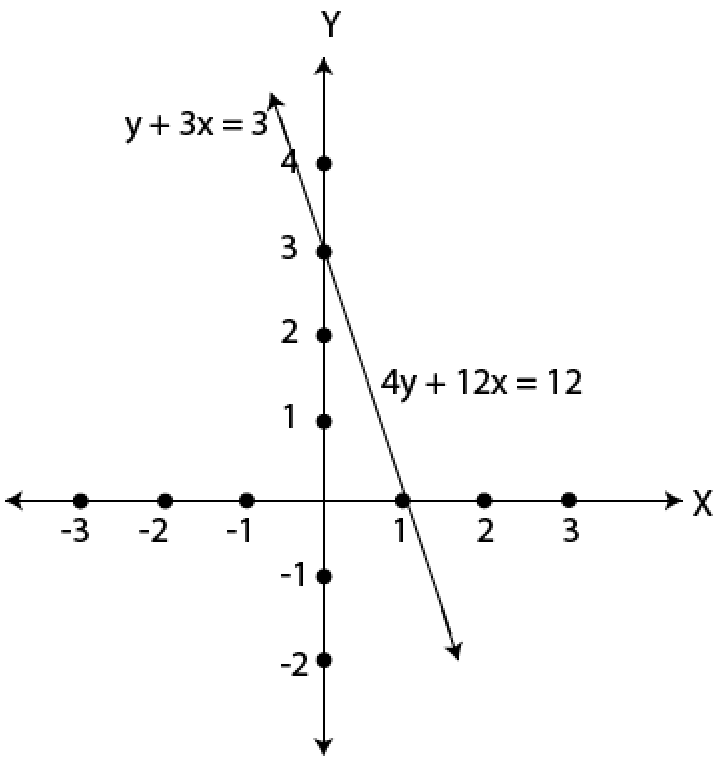
There are equations that the same gradient (inclination) and the same x and y intercept. These system of equation have all of their points coinciding, as such we say they have infinitely many solutions.

Example 11.2

Solve the system of equations below graphically $\begin{cases} y + 3x = 3 \\ 4y + 12x = 12 \end{cases}$

Solution

The x and y intercepts of the equations above are 1 and 3 respectively is shown in figure 11.2.

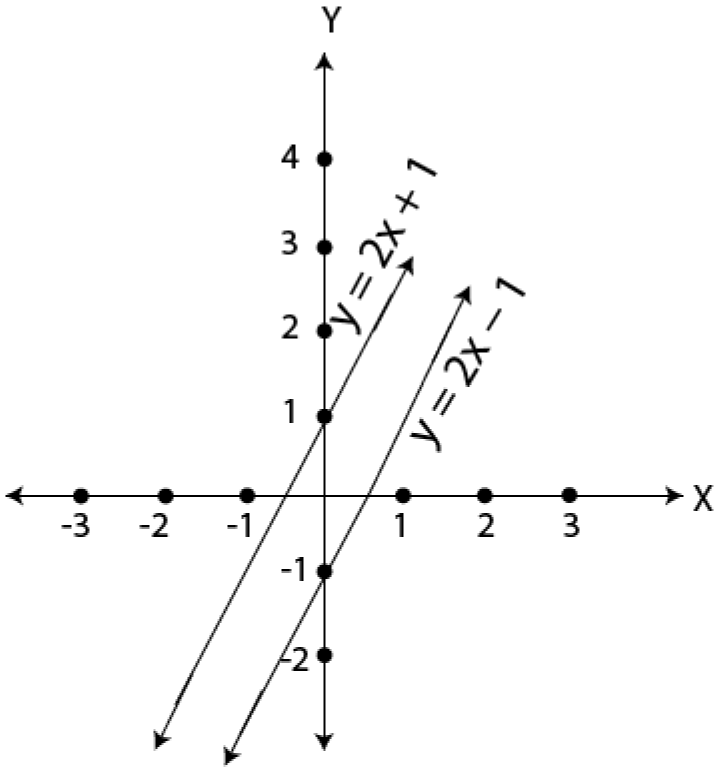


Hence, the solution set is ($\{x, y): (x, 3 - 3x)\}$

Note that, any system of linear equations which has infinitely many solution is said to be INCONSISTENT and DEPENDENT.

Example 11.3

Solve the following system of equations graphically: $\begin{Bmatrix} y = 2x - 1 \\ y = 2x + 1 \end{Bmatrix}$



Shows two straight lines that are not intersecting however much they are neither side. We call them parallel lines. I or the reason these two lines are not the system does not have a solution. Any system of linear equations that doesn't have a solution is said to be INCONSISTENT.

Table 11.3 summarizing the study on graphical solutions of linear equation

Graph	Inclination (gradient)	Name of the system
Lines that intersect	Distinct inclinations and distinct intercepts.	Consistent and independent.
Parallel lines	Same inclinations but distinct intercepts.	Inconsistent.
Lines that concide	Same inclinations and intercept.	Consistent and Dependent

1. The following system of linear equations graphically and state whether the consistent and independent, inconsistent or consistent and dependent:

2. $\begin{cases} x + y = 8 \\ 2x + 3y = 18 \end{cases}$

3. $\begin{cases} x - y = 8 \\ x - y = 4 \end{cases}$

4. $\begin{cases} 2x + 3y = -3 \\ -3x + y = -5 \end{cases}$

5. $\begin{cases} 2x + y = 6 \\ 6x + 3y = 18 \end{cases}$

6. Write an equation the line that is inconsistent with the line 3x – y – 1 =0 and goes though (1, 1).

7. Write an equation of the line is inconsistent with the line 3x – y – 1 = 0 and

Goes through (1, 1)

8. Line kx + ty – 8 is consistent and dependent to line 12x + 15y = 24. Find the values of k and t that fit the stated condition.

Graphing systems of inequalities

Systems of inequalities can also be solved graphically. They have solutions like linear equations, but their solution are regions which satisfy set of inequalities being graphed.

1. Obtain the line of equality (boundary lines) by replacing the inequality sign with an equal sign.

2. Draw the line of equality (equalities) in Cartesian coordinate plane.

3. Test for the region satisfies the inequality (inequalities) and shade it

Example **11.4**

Solve the following system graphically $\begin{cases} y \leq 4 \\ x \geq -3 \\ y > x+1 \end{cases}$

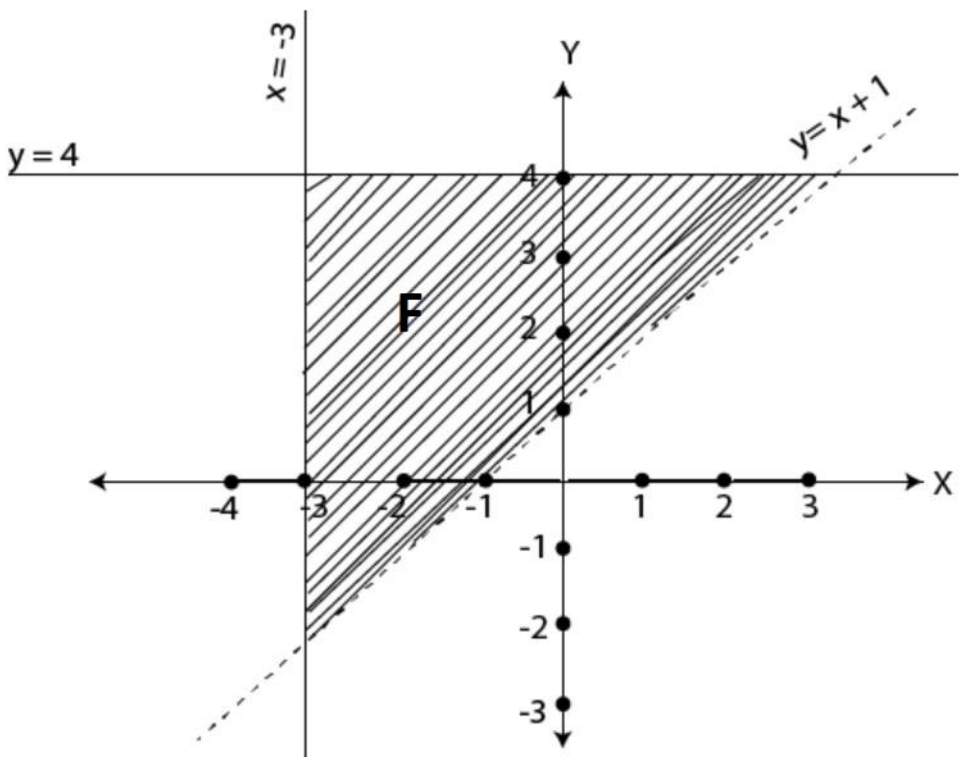
Solution

The line of equalities for the above inequalities are y = 4, y = -3 and y = x + 1

Note

If the line of equality was derived from the inequality ≥ or ≤ the line must be drawn as full line, but if it was from > or < the line must be made broken

Draw each of the above lines in the same Cartesian coordinate plane, test and shade the required region.



The figure shows the required graph for the set of the inequalities.

The region marked F in figure 11.4 is the solution of the set of the given inequalities. We call it feasible region. If the system of inequalities does not have a region that satisfies all the inequalities in a given set, that system does not have a solution.

Exercise 11.2

Does the point given against each system of inequalities satisfy the system?

1. $\begin{cases} y < x + 2 \\ y > 4x - 2 \end{cases}$ (0, 0)

2. $\begin{cases} y < x \\ y > 2x + 1 \end{cases}$ (1, 0)

3. $\begin{cases} y < 8 \\ x \geq -3 \end{cases}$ (5, 5)

4. $\begin{cases} y + 2x > 4 \\ x > 6 \end{cases}$ (0, 4)

5. $\begin{cases} x > 1 \\ y \leq 5 \\ y > x + 2 \end{cases}$ (1, 1)

Solve each of the following inequalities graphically

6. $\begin{cases} y > 3 \\ x \leq 2 \end{cases}$

7. $\begin{cases} y \leq 4 \\ x \geq 1 \end{cases}$

8. $\begin{cases} x > 3 \\ y + 2x > 2 \end{cases}$

9. $\begin{cases} x < -1 \\ x - y > 1 \end{cases}$

10. $\begin{cases} y \geq x - 2 \\ y \leq \frac{1}{2}x + 2 \end{cases}$

11. $\begin{cases} |x - 3| < 2 \\ x \leq 4 \end{cases}$

12. $\begin{cases} y \geq 1 \\ x \leq 2 \\ y > x \end{cases}$

Linear programming (Linear Optimization)

Linear programming (optimization) is the youngest branch of mathematics which was developed by George Dantizig in 1940's. It is the model that consist of methods for solving many day to day life problems that in one way or the other demand the optimization of resources which are scarce.

The actual fact, linear programming is a mathematical procedure which helps to allocate, select, schedule and evaluate limited resources in the best possible way . These limited resources in this case may be money in the form of capital or costs. They can also be raw materials, labour and production equipments.

Generally, linear programming problems always are such that they tend to fulfill a linear statement of the form, mx + ny, called objective function. So if $a_1x + b_1y \leq c_1$ and $a_2x + b_2yc_2$ are the only constraints that restrict the maximization of problems and if also $x \geq 0$ and are their non – negativity constraints, then the standard way of reporting (writing) this problem is

Maximize: $m_1x + n_1y$

Subject to: $a_1x + b_1y \leq c_1$

$a_2x + b_2y \leq c_2$

$x \geq 0$

$y \geq 0$

Similarly, if the problem is not of minimization of the objective function $m_5x \div n_5y$ and $a_3x + b_3y \geq c_3$ while $a_4x + b_4y \geq c_4$, $x \geq 0$ and $y \geq 0$ are bounding constraints, then standard form is:

Minimize: $m_5x + n_5y$

Subject to: $a_3x + b_3y \geq c_3$

$a_4x + b_4y \geq c_4$

$x \geq 0$

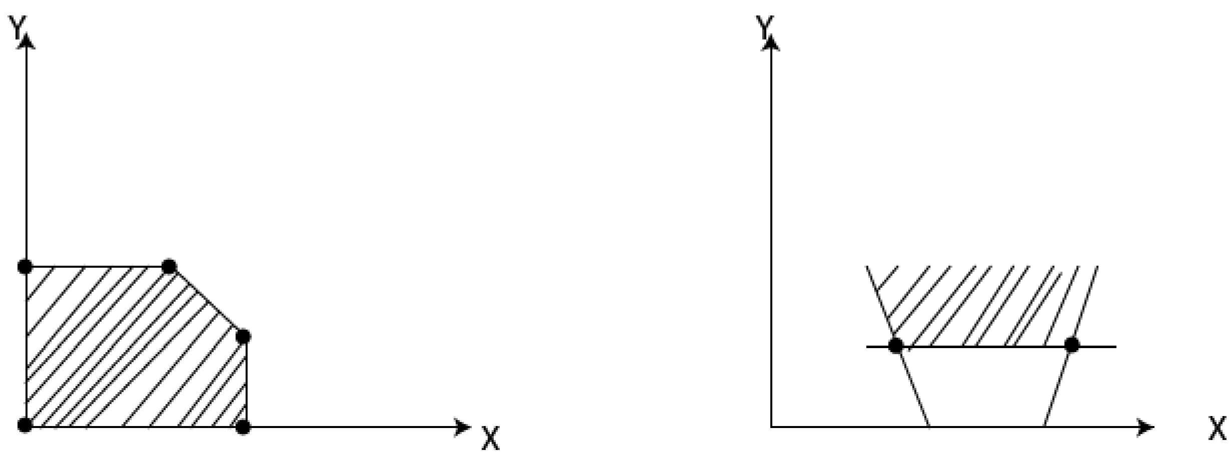
$y \geq 0$

Note:

1. An objective function is a mathematical statement that defines how best a data fits a particular assertion. It describes the essential characteristics of the alternative

2. Constraints are sets of inequalities for a linear programming problem in question, i.e $a_1x + b_1y \leq c_1$, $a_2x + b_2y \leq c_2$, $x \geq 0$ and $y \geq 0$ in the standard form for maximization given earlier.

- If (x, y) happens to be one of the point that satisfies all the constraints, we call it a feasible solution. A set of all feasible solution form what we call a feasible region. The feasible region includes the boundary lines (lines of equality). If the feasible region is enclosed by a polygon, it is said to be bounded or unbounded (see figure 11.5).



If a feasible region has no point, the constraints involved are inconsistent and the problem does not have solution. The solution of any linear programming problem is given by one (some) of the feasible solution (s).However, these feasible solutions are many, which one should be taken as optimum solution? The following answer that

THEOREM

For a feasible region has no point of a linear programming which is bounded and not empty, the objective function attains either a maximum or a minimum value at the corner points (vertices) of the feasible region. If the feasible is unbounded, the objective function may or may not attain a minimum or maximum value, but if it attain, it does so at the corner points (vertices).

According to the theorem, the optimum solution comes the corner points, ie once you have graphed the constraints in Cartesian coordinate plane, test the corner points objective function, and then identify the optimum solution.

Example 11. 5

Siwatu wants to earn at least 13000/= this week. Her father has agreed to pay her 5000/= to know the lawn and 2000/= to weed the garden in a day suppose Siwatu mows the lawn once, what minimum number of days will she have to spend weeding the garden?

Solution

Let x and y be the number of days Siwatu spends moving the lawn and wedding the garden respectively.

Objective function $x + y$

For minimization

Subject to: $5000x + 2000y \geq 13000$

$5x + 2y \geq 13$

Hence, she mows the lawns once, then

$x \geq 1$ and $x \geq 0$ $y \geq 0$

Linear programming problems associated with this question is as follows

Minimize: x + y

Object to: 5x + 2y ≥ 13

x ≥ 1

x ≥ 0

y ≥ 0

The feasible region of the problem, together with the corner points are shown in figure the feasible region is indicated by shading.

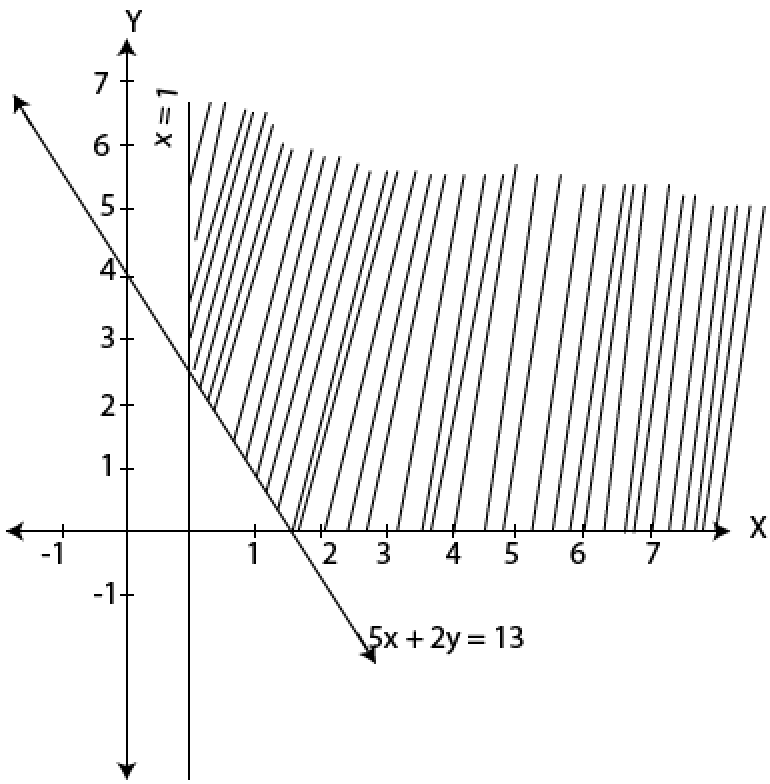


Table 11.4

Corner points	x + y
(1, 4)	5
(2.6, 0)	2. 6

(5) Is the optimum solution row.

From table 11.4, the optimum solution is (1,4) meaning that to earn the required amount of money, Siwatu will have to work 4 days a week.

Example 11.6

A carpenter makes two kinds of furniture, i.e chairs and tables. Two operations M and N are used. Operation M is limited to 20 days a month while operations N are limited to 15 days per month. Table 11.5 shows the amount of time each operation takes for one chair and one table and the profit it makes on each item.

Table 11.5

Furniture	Operations M	Operations N	Profit
Chair	2 day	3 days	20000/=
Table	4 days	1 day	24000/=

How many tables and chairs should the carpenter make in a month to maximize the income?

Solution

Let the number of chairs and tables be x and y respectively

Object function:

$f(x,y) = 20000x + 24000y$
$2000x + 24000y$

$2x + 4y \leq 20$

$x + 2y \leq 20$

Also $3x + y \leq 15$

$x \geq 0$ and $y \geq 0$

Thus, the linear programming problems associated with this question is as follow

Maximize: 20000x + 24000y

Subject to: $x + 2y \leq 10$

$3x + y \leq 15$

$x \geq 0$

$y \geq 0$

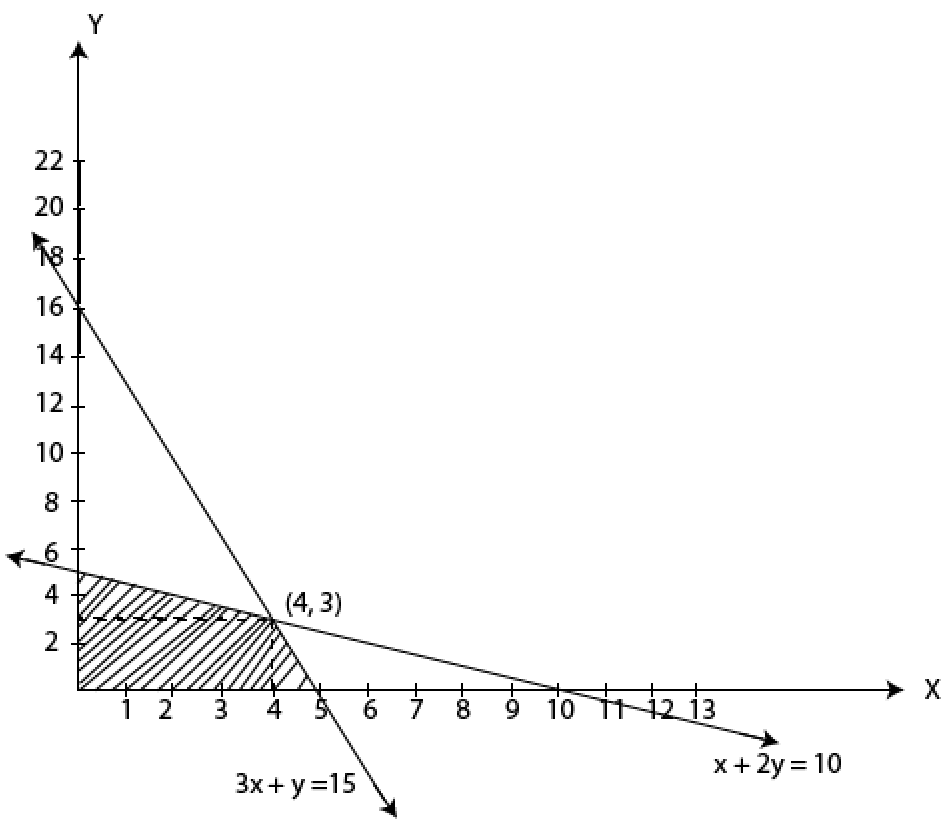


Figure shows the graph of linear programming problem given in example 11.6

Table 11.6

Corner points	20000x + 24000y
(0, 0)	0
(0, 5)	120,000

(4, 3)	152,000
(5, 0)	100,000

152,000 is maximum value optimum solution row.

To maximize the income a month, a carpenter should make 4 chairs and 3 tables

Exercise 11.3

Answer each of the following questions as directed.

1. a) Are the solutions of linear programming problems unique?

   Explain

   b) What do you understand by the terms i) Feasible solutions? ii) Feasible region? iii) Constraints? iv) Objective function?

2. Maximize: $30x + 50y$

   Subject to: $2x + 8y \leq 60$

   $4x + 4y \leq 60$,$x \geq 0$

   $x \geq 0$

3. Maximize: $x + 6y$

   Subject to: $5x + 10y \leq 50$

   $x + y \leq 10$

   $x \geq 0$

   $y \geq 0$

4. Minimize: $60 – (x + y)$

   Subject to: $x + y \geq 3$

   $x \leq 4$

$x + 3y \geq 15$

$x \geq 0$

$y \geq 0$

5. Maximize a daily production of Mandazi and Chapati of mama Masawe's business which is mixture of items M_1, and M_2. The required information is summarized in table 11.7

Table 11.7

Item	**production**		**Daily supply**
	Maandazi	chapati	
M1	5	2.5	10kg
M2	5	7.5	15kg
Net profit in TZ shs	300	250	

6. A farmer produces and sells onions and tomatoes. His profit being 150000/=and 100000/= respectively. His production involves two workers, M_1 and M_2 who are available for 100 days and 80 days a year respectively. M_1 works in union farms for 6 days in tomatoes farm for 10 days. M_2 works in onion farm for 4 days and in tomatoes farms for 4 days determine production costs that will maximize the production.

7. Mnazimmoja parking lot has a total area of $600M^2$.A car requires 6 M^2 and a bus 15 M^2 of space. The attendants can handle not more than 60 vehicles. If car is charged 2500/= and a bus 7500/=, how many of each should be accepted to maximize the profits? What is the of space left unused?

8. A bus contractor is contracted to transport 500 workers to their working places.For the contract, he has 3 type M buses capable of carrying 50 workers and 4 type N buses capable of carrying 85 workers each. Only five drivers are available at the moment. If no bus is to

repeat, how will the cost be reduced given that running type M bus costs 5000/= and type N bus costs 4000/=? How many drivers should be used?

9. Rose is a shopkeeper of Hilanyeupe. Shop in one occasion, she accepted goods with delivery note of a number of beef cans which was not visible enough. She trusted the supplier so she did not look at the delivery note and she did not bother herself counting. Two days later, she took the delivery note for the beef supplied two days ago and she found that the number was a three digits one with 4 as the middle digit. Owing to this, she concluded that the first digit can be less or equal to six (6) and the third digit can be smaller or equal to 6, and that the sum of the two unknown digits cannot exceed 8. If the difference of 6 times the first digit and the third'' is to be big enough, what was the approximate number? (Assuming her conclusion was right)

Revision exercise 11

Solve each of the following graphically stating whether the system is inconsistent, consistent and independent or consistent and dependent.

1. $\begin{cases} x + y = 7 \\ 2x - y = 5 \end{cases}$

2. $\begin{cases} x + y = -1 \\ 5x - 2y = 16 \end{cases}$

3. $\begin{cases} 7x - 12y = 11 \\ x + 3y = 11 \end{cases}$

4. $\begin{cases} y = \frac{1}{2}x - 8 \\ y = \frac{1}{2}x + 6 \end{cases}$

5. $\begin{cases} x + 4y = 8 \\ 3x + 12y - 24 = 0 \end{cases}$

Solving each of the system of linear inequalities that follows

6. $\begin{cases} 2x + y < 7 \\ x + 2y > 3 \end{cases}$

7. $\begin{cases} x < 4 \\ y < 3 \\ y < x \\ y > 0 \end{cases}$

8. $\begin{cases} y < 2 \\ x - y > 2 \end{cases}$

9. $\begin{cases} x + 2y > 5 \\ x - y > 6 \\ \end{cases}$

10. $\begin{cases} 2x + 3y < 8 \\ x - y < 2 \end{cases}$

11. $\begin{cases} 3y - 2x \leq 5 \\ x + 4y \geq 14 \end{cases}$

12. $\begin{cases} x > 0 \\ y > 0 \\ y < 5 \\ x > 4 \end{cases}$

13. Define the terms

    i) Linear programming

    ii) Bounded linear programming problem

    iii) Unbounded linear programming problem

14. Maximize: x + 5y

    Subject to:

15. Minimize: 3x + 2y

    Subject to: $0 \leq x \leq 8$

    $0 \leq y \leq 7$

$x + y \leq 10$

16. Maximize $10x + 20y$

Subject to: $0 < x < 9$

$y > 18$

$x + 3y \leq 75$

17. Maximize: $2500x + 7500$

Subject to: $x + y \leq 60$

$x + 5y \leq 100$

$x \geq 0, \ y \geq 0$

18. Maximize: $120x + 250y$

Subject to: $0 \leq x \leq 300$

$0 \leq y \leq 200$

$2x + 3y \leq 1200$

19.A store manager of a certain company finds that he can store a batch of 3 machines in 2 m^2 and batch of 4 freezers in $4m^2$. He reserved $160m^2$ altogether for this section of his stock and never allows the number of items to exceed 400. Find the greatest number of items he can accept subject to those conditions? If the machine is sold at 450000/= and a freezer at 225000/=, find the combination of items he should keep to maintain the greatest possible value of stock.

20.A company is intending to buy two equipments, E_1 and E_2, these equipments are to be used for a certain production. More information of these equipments is given in table 11.8

Table 11.

Equipment	**Output per hour**	**Profit per hour**	**Floor space take**
E_1	45	60000/=	$5m^2$

E_2	30	37500/=	$4m^2$

The company is prepared to buy at least E_2 equipment as E_1 equipments. At least 360 products must be produced per hour and up to $80m^2$ of the floor space is available find the combination of the equipments that must be bought to maximize the profit will the floor space be left? If yes how much?

21. In a physics examination consisting of paper 1 (p_1) and paper 2 (p_2), both papers were marked out of 100. A candidate is given a mark x for p_1 and mark y for p_2. Someone is considered to have passed if 3x + 5y is at least 210, but candidates must score above 35 marks on p_1 and above 44 marks on p_2, Find the lowest value of x + y for any candidates who passes and give the corresponding values of x and y.